Cylindrical coordinates in action

Lesson 27 · Module 2 — Multiple Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$, the plane $z = 0$, and the plane $z = x + 4$.
2. Calculate the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and inside the cylinder $x^2 + y^2 = 1$.
3. Find the volume of the solid bounded by the cone $z = \sqrt{3(x^2 + y^2)}$ and the sphere $x^2+y^2+z^2 = 4$.

Solutions

1. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$, the plane $z = 0$, and the plane $z = x + 4$.

   - Convert to cylindrical: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
   - Bounds: $0 \le z \le r\cos\theta + 4$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$.
   - Integral: $\int_0^{2\pi}\int_0^3\int_0^{r\cos\theta+4} r\,dz\,dr\,d\theta$.
   - Inner: $\int_0^{r\cos\theta+4} r\,dz = r(r\cos\theta+4) = r^2\cos\theta + 4r$.
   - Middle: $\int_0^3 (r^2\cos\theta + 4r)\,dr = [\frac{1}{3}r^3\cos\theta + 2r^2]_0^3 = 9\cos\theta + 18$.
   - Outer: $\int_0^{2\pi}(9\cos\theta + 18)\,d\theta = [9\sin\theta + 18\theta]_0^{2\pi} = 36\pi$.

   Answer: 36\pi

2. Calculate the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and inside the cylinder $x^2 + y^2 = 1$.

   - The sphere is $z^2 = 4 - r^2$, so $z = \pm\sqrt{4-r^2}$.
   - The cylinder gives $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
   - Integral: $\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$.
   - Inner: $2r\sqrt{4-r^2}$.
   - Middle: $\int_0^1 2r\sqrt{4-r^2}\,dr$. Let $u = 4-r^2, du = -2rdr$. $\int_4^3 -\sqrt{u}\,du = [\frac{2}{3}u^{3/2}]_3^4 = \frac{2}{3}(8-3\sqrt{3})$.
   - Outer: $2\pi \cdot \frac{2}{3}(8-3\sqrt{3}) = \frac{4\pi}{3}(8-3\sqrt{3})$.

   Answer: \frac4\pi3(8 - 3\sqrt3)

3. Find the volume of the solid bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the sphere $x^2+y^2+z^2 = 4$.

   - Cone: $z = \sqrt{3}r$. Sphere: $z = \sqrt{4-r^2}$.
   - Intersection: $\sqrt{3}r = \sqrt{4-r^2} \implies 3r^2 = 4 - r^2 \implies 4r^2 = 4 \implies r = 1$.
   - Bounds: $0 \le \theta \le 2\pi$, $0 \le r \le 1$, $\sqrt{3}r \le z \le \sqrt{4-r^2}$.
   - Integral: $\int_0^{2\pi}\int_0^1\int_{\sqrt{3}r}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$.
   - Inner: $r(\sqrt{4-r^2} - \sqrt{3}r) = r\sqrt{4-r^2} - \sqrt{3}r^2$.
   - Middle: $\int_0^1 (r\sqrt{4-r^2} - \sqrt{3}r^2)\,dr = [-\frac{1}{3}(4-r^2)^{3/2} - \frac{\sqrt{3}}{3}r^3]_0^1 = (-\frac{1}{3}(3\sqrt{3}) - \frac{\sqrt{3}}{3}) - (-\frac{1}{3}(8) - 0) = -\sqrt{3} - \frac{\sqrt{3}}{3} + \frac{8}{3} = \frac{8-4\sqrt{3}}{3}$.
   - Outer: $2\pi \cdot \frac{8-4\sqrt{3}}{3} = \frac{8\pi(2-\sqrt{3})}{3}$.

   Answer: \frac8\pi(2 - \sqrt3)3