Applications: mass, centroids, moments of inertia

Lesson 25 · Module 2 — Multiple Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Find the mass of a semi-circular lamina $R = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le \pi\}$ with density $\rho(x, y) = x^2 + y^2$.
2. A rectangular lamina occupies $0 \le x \le 1, 0 \le y \le 2$ with density $\rho(x, y) = x$. Find the center of mass $(\bar{x}, \bar{y})$.
3. Find the moment of inertia I_z for a disk of radius 1 with constant density $\rho = 1$.

Solutions

1. Find the mass of a semi-circular lamina $R = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le \pi\}$ with density $\rho(x, y) = x^2 + y^2$.

    - Convert density to polar: $\rho = r^2$.
    - Set up the integral: $m = \int_0^\pi \int_0^1 r^2 \cdot r \, dr \, d\theta$.
    - Integrate r^3 to get $r^4/4$ from 0 to 1, which is 1/4.
    - Integrate 1/4 with respect to θ from 0 to π.

    Answer: \frac\pi4

2. A rectangular lamina occupies $0 \le x \le 1, 0 \le y \le 2$ with density $\rho(x, y) = x$. Find the center of mass $(\bar{x}, \bar{y})$.

    - Compute mass: $m = \int_0^1 \int_0^2 x \, dy \, dx = \int_0^1 2x \, dx = 1$.
    - Compute M_y: $\int_0^1 \int_0^2 x(x) \, dy \, dx = \int_0^1 2x^2 \, dx = 2/3$.
    - Compute M_x: $\int_0^1 \int_0^2 y(x) \, dy \, dx = \int_0^1 [y^2/2]_0^2 x \, dx = \int_0^1 2x \, dx = 1$.
    - Calculate coordinates: $\bar{x} = (2/3)/1 = 2/3$ and $\bar{y} = 1/1 = 1$.

    Answer: (2/3, 1)

3. Find the moment of inertia I_z for a disk of radius 1 with constant density $\rho = 1$.

    - Use polar coordinates: $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta$.
    - Set up integral: $I_z = \int_0^{2\pi} \int_0^1 r^2 \cdot 1 \cdot r \, dr \, d\theta$.
    - Integrate r^3 to get $r^4/4$ from 0 to 1, which is 1/4.
    - Integrate 1/4 from 0 to 2π.

    Answer: \frac\pi2
