Double integrals in polar coordinates — deepened

Lesson 24 · Module 2 — Multiple Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Evaluate the double integral of $f(x, y) = x^2 + y^2$ over the region R in the first quadrant bounded by the circle $x^2 + y^2 = 9$.
2. Find the volume under the surface $z = e^{-(x^2+y^2)}$ over the region R which is the upper half-disk of radius 1.
3. Evaluate $\iint_R y \, dA$ where R is the region inside the circle $r = 2\sin\theta$.

Solutions

1. Evaluate the double integral of $f(x, y) = x^2 + y^2$ over the region R in the first quadrant bounded by the circle $x^2 + y^2 = 9$.

   - Convert to polar: $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$.
   - Set bounds: r goes from 0 to 3, and θ goes from 0 to $\pi/2$.
   - Set up integral: $\int_0^{\pi/2} \int_0^3 r^2 \cdot r\,dr\,d\theta$.
   - Integrate r^3: $[\frac{1}{4}r^4]_0^3 = \frac{81}{4}$.
   - Integrate with respect to θ: $\int_0^{\pi/2} \frac{81}{4}\,d\theta = \frac{81\pi}{8}$.

   Answer: $\frac{81\pi}{8}$

2. Find the volume under the surface $z = e^{-(x^2+y^2)}$ over the region R which is the upper half-disk of radius 1.

   - Convert to polar: $z = e^{-r^2}$ and $dA = r\,dr\,d\theta$.
   - Set bounds: r goes from 0 to 1, and θ goes from 0 to π.
   - Set up integral: $\int_0^{\pi} \int_0^1 e^{-r^2} r\,dr\,d\theta$.
   - Use u-substitution $u = -r^2$, $du = -2r\,dr$: $\int_0^1 re^{-r^2}\,dr = [-\frac{1}{2}e^{-r^2}]_0^1 = \frac{1}{2}(1 - e^{-1})$.
   - Integrate with respect to θ: $\int_0^{\pi} \frac{1}{2}(1 - e^{-1})\,d\theta = \frac{\pi}{2}(1 - e^{-1})$.

   Answer: $\frac{\pi}{2}(1 - e^{-1})$

3. Evaluate $\iint_R y\,dA$ where R is the region inside the circle $r = 2\sin\theta$.

   - The region $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius 1.
   - Bounds: r goes from 0 to $2\sin\theta$, and θ goes from 0 to π.
   - Convert y to polar: $y = r\sin\theta$.
   - Set up integral: $\int_0^{\pi} \int_0^{2\sin\theta} (r\sin\theta) r\,dr\,d\theta$.
   - Integrate r^2: $[\frac{1}{3}r^3]_0^{2\sin\theta} = \frac{8}{3}\sin^3\theta \cdot \sin\theta = \frac{8}{3}\sin^4\theta$.
   - Integrate $\int_0^{\pi} \frac{8}{3}\sin^4\theta\,d\theta$ using power reduction: $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.
   - Evaluating from 0 to π: $\frac{8}{3}[\frac{3}{8}\theta]_0^{\pi} = \pi$.

   Answer: π
