

Double integrals — volume, mass, and the meaning

Lesson 21 · Module 2 — Multiple Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Compute the double integral of $f(x, y) = x^2 + y^2$ over the rectangle $R = [0, 1] \times [0, 1]$.
2. Find the volume under the surface $z = 4 - x - y$ over the region R bounded by $x = 0$, $y = 0$, and $x + y = 1$.
3. A thin plate occupies the region R bounded by $y = x$ and $y = x^2$. The density is $\rho(x, y) = x$. Find the total mass.

Solutions

1. Compute the double integral of $f(x, y) = x^2 + y^2$ over the rectangle $R = [0, 1] \times [0, 1]$.

- Set up the iterated integral: $\int_0^1 \int_0^1 (x^2 + y^2) dy dx$.
- Integrate with respect to y : $[x^2 y + \frac{1}{3} y^3]_0^1 = x^2 + \frac{1}{3}$.
- Integrate the result with respect to x : $\int_0^1 (x^2 + \frac{1}{3}) dx = [\frac{1}{3} x^3 + \frac{1}{3} x]_0^1$.
- Evaluate: $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

Answer: $2/3$

2. Find the volume under the surface $z = 4 - x - y$ over the region R bounded by $x = 0$, $y = 0$, and $x + y = 1$.

- Set up the integral with y limits from 0 to $1 - x$ and x limits from 0 to 1: $\int_0^1 \int_0^{1-x} (4 - x - y) dy dx$.
- Inner integral: $[4y - xy - \frac{1}{2} y^2]_0^{1-x} = 4(1-x) - x(1-x) - \frac{1}{2}(1-x)^2$.
- Simplify the integrand: $4 - 4x - x + x^2 - \frac{1}{2}(1 - 2x + x^2) = 3.5 - 4x + 0.5x^2$.
- Outer integral: $\int_0^1 (3.5 - 4x + 0.5x^2) dx = [3.5x - 2x^2 + \frac{1}{6} x^3]_0^1$.
- Evaluate: $3.5 - 2 + \frac{1}{6} = 1.5 + \frac{1}{6} = \frac{3}{2} + \frac{1}{6} = \frac{9+1}{6} = \frac{10}{6} = \frac{5}{3}$.

Answer: $5/3$

3. A thin plate occupies the region R bounded by $y = x$ and $y = x^2$. The density is $\rho(x, y) = x$. Find the total mass.

- The curves intersect at $x = 0$ and $x = 1$. The line $y = x$ is the upper boundary.
- Set up the integral: $\int_0^1 \int_{x^2}^x x dy dx$.
- Inner integral: $[xy]_{x^2}^x = x(x) - x(x^2) = x^2 - x^3$.
- Outer integral: $\int_0^1 (x^2 - x^3) dx = [\frac{1}{3} x^3 - \frac{1}{4} x^4]_0^1$.
- Evaluate: $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Answer: $1/12$