Module 1 review + self-test

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Practice problems

1. Find the directional derivative of $f(x, y) = e^{xy} \cos(x + y)$ at the point $(0, 0)$ in the direction of $\mathbf{v} = \langle 1, 1 \rangle$.
2. Find and classify the critical points of $f(x, y) = x^2 + 2y^2 - 2xy - 4x$.
3. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

Solutions

1. Find the directional derivative of $f(x, y) = e^{xy}\cos(x + y)$ at the point $(0, 0)$ in the direction of $\mathbf{v} = \langle 1, 1\rangle$.

   - Compute partial derivatives: $f_x = ye^{xy}\cos(x + y) - e^{xy}\sin(x + y)$ and $f_y = xe^{xy}\cos(x + y) - e^{xy}\sin(x + y)$.
   - Evaluate at $(0, 0)$: $f_x(0, 0) = 0(1)(1) - (1)(0) = 0$ and $f_y(0, 0) = 0(1)(1) - (1)(0) = 0$.
   - The gradient is $\nabla f(0, 0) = \langle 0, 0\rangle$.
   - The directional derivative is the dot product of the gradient and the unit vector $\mathbf{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$.
   - Result: $0(1/\sqrt{2}) + 0(1/\sqrt{2}) = 0$.

   Answer: 0

2. Find and classify the critical points of $f(x, y) = x^2 + 2y^2 - 2xy - 4x$.

   - Find partials: $f_x = 2x - 2y - 4$ and $f_y = 4y - 2x$.
   - Set to zero: $2x - 2y = 4$ and $2x = 4y \implies x = 2y$.
   - Substitute $x = 2y$ into first eq: $2(2y) - 2y = 4 \implies 2y = 4 \implies y = 2$. Then $x = 4$.
   - Critical point is $(4, 2)$.
   - Compute Hessian: $f_{xx} = 2, f_{yy} = 4, f_{xy} = -2$.
   - Determinant $D = (2)(4) - (-2)^2 = 8 - 4 = 4$.
   - Since $D > 0$ and $f_{xx} > 0$, it is a local minimum.

   Answer: Local minimum at (4, 2)

3. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

   - Set up equations: $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x, 2y\rangle$.
   - System: $y = 2\lambda x$ and $x = 2\lambda y$.
   - Substitute y: $x = 2\lambda(2\lambda x) = 4\lambda^2 x$.
   - Either $x = 0$ (which implies $y = 0$, but $(0, 0)$ is not on the circle) or $4\lambda^2 = 1 \implies \lambda = \pm 1/2$.
   - If $\lambda = 1/2$, $y = x$. Constraint $x^2 + x^2 = 1 \implies 2x^2 = 1 \implies x = \pm 1/\sqrt{2}$. Points: $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. $f = 1/2$.
   - If $\lambda = -1/2$, $y = -x$. Constraint $x^2 + (-x)^2 = 1 \implies 2x^2 = 1 \implies x = \pm 1/\sqrt{2}$. Points: $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. $f = -1/2$.

   Answer: Max value 1/2, Min value -1/2