Lagrange Multipliers — Deepened

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Practice problems

1. Find the maximum and minimum values of $f(x, y) = x+2y$ subject to the constraint $x^2+y^2 = 5$.
2. Find the point on the line $y = 2x + 3$ that is closest to the origin $(0, 0)$.
3. Maximize $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 12$ for $x, y, z > 0$.

Solutions

1. Find the maximum and minimum values of $f(x, y) = x+2y$ subject to the constraint $x^2+y^2 = 5$.

   - Set up the gradients: $\nabla f = (1, 2)$ and $\nabla g = (2x, 2y)$.
   - Apply Lagrange equations: $1 = \lambda(2x)$ and $2 = \lambda(2y)$.
   - Divide the equations: $\frac{1}{2} = \frac{2x}{2y} \implies y = 2x$.
   - Substitute into constraint: $x^2 + (2x)^2 = 5 \implies 5x^2 = 5 \implies x = \pm 1$.
   - Find points: $(1, 2)$ and $(-1, -2)$.
   - Evaluate f: $f(1, 2) = 5$ and $f(-1, -2) = -5$.

   Answer: Max is 5, Min is -5

2. Find the point on the line $y = 2x + 3$ that is closest to the origin $(0, 0)$.

   - Minimize distance squared: $f(x, y) = x^2 + y^2$ subject to $g(x, y) = 2x - y = -3$.
   - Gradients: $\nabla f = (2x, 2y)$ and $\nabla g = (2, -1)$.
   - Equations: $2x = 2\lambda$ and $2y = -\lambda$.
   - Express in terms of λ: $x = \lambda$ and $y = -\lambda/2$.
   - Substitute into constraint: $2(\lambda) - (-\lambda/2) = -3 \implies 2.5\lambda = -3 \implies \lambda = -1.2$.
   - Find coordinates: $x = -1.2$, $y = 0.6$.

   Answer: (-1.2, 0.6)

3. Maximize $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 12$ for $x, y, z > 0$.

   - Gradients: $\nabla f = (yz, xz, xy)$ and $\nabla g = (1, 1, 1)$.
   - Equations: $yz = \lambda$, $xz = \lambda$, $xy = \lambda$.
   - Equating them: $yz = xz \implies y = x$ (since $z > 0$) and $xz = xy \implies z = y$.
   - Thus $x = y = z$.
   - Substitute into constraint: $x + x + x = 12 \implies 3x = 12 \implies x = 4$.
   - Point is $(4, 4, 4)$, and $f(4, 4, 4) = 64$.

   Answer: 64
