Optimisation clinic

Lesson 18 · Module 1 — Differentiation in Several Variables · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2$ on the region $x^2 + y^2 \leq 1$.
2. Find the absolute extrema of $f(x, y) = 2x + 3y$ on the rectangle $0 \leq x \leq 2, 0 \leq y \leq 1$.
3. A rectangular box with a square base and no top must have a volume of 32 cubic metres. Find the dimensions that minimise the surface area.

Solutions

1. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2$ on the region $x^2 + y^2 \leq 1$.

   - Find interior critical points: $f_x = 2x = 0$ and $f_y = 2y = 0$, so $(0, 0)$ is a critical point. $f(0, 0) = 0$.
   - Check the boundary $x^2 + y^2 = 1$. On this boundary, $f(x, y) = 1$ for all points.
   - Compare values: the minimum is 0 at $(0, 0)$ and the maximum is 1 at any point on the boundary circle.

   Answer: Min: 0, Max: 1

2. Find the absolute extrema of $f(x, y) = 2x + 3y$ on the rectangle $0 \leq x \leq 2, 0 \leq y \leq 1$.

   - Interior: $f_x = 2, f_y = 3$. No critical points since the gradient is never zero.
   - Boundary $x = 0$: $f(0, y) = 3y$, max 3 at $y = 1$, min 0 at $y = 0$.
   - Boundary $x = 2$: $f(2, y) = 4 + 3y$, max 7 at $y = 1$, min 4 at $y = 0$.
   - Boundary $y = 0$: $f(x, 0) = 2x$, max 4 at $x = 2$, min 0 at $x = 0$.
   - Boundary $y = 1$: $f(x, 1) = 2x + 3$, max 7 at $x = 2$, min 3 at $x = 0$.
   - Comparing all: absolute min is 0 at $(0, 0)$ and absolute max is 7 at $(2, 1)$.

   Answer: Min: 0, Max: 7

3. A rectangular box with a square base and no top must have a volume of 32 cubic metres. Find the dimensions that minimise the surface area.

   - Let x be the side of the square base and h be the height. Volume $V = x^2h = 32$, so $h = 32/x^2$.
   - Surface area $S = x^2 + 4xh$. Substitute h: $S(x) = x^2 + 4x(32/x^2) = x^2 + 128/x$.
   - Find critical points: $S'(x) = 2x - 128/x^2 = 0$.
   - Solve $2x^3 = 128 \implies x^3 = 64 \implies x = 4$.
   - Find h: $h = 32/(4^2) = 32/16 = 2$.
   - Check second derivative: $S''(x) = 2 + 256/x^3$, which is positive for $x = 4$, confirming a minimum.

   Answer: Base side 4m, Height 2m
