Critical points and the second derivative test

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Practice problems

1. Find and classify the critical points of $f(x, y) = x^2 + 2y^2 + xy$.
2. Classify the critical point at $(-1, 0)$ for the function $f(x, y) = x^2 - 3y^2 + 2x$.
3. Find and classify all critical points of $f(x, y) = x^3 - 3x + y^2$.

Solutions

1. Find and classify the critical points of $f(x, y) = x^2 + 2y^2 + xy$.

    - Compute partials: $f_x = 2x + y$ and $f_y = 4y + x$.
    - Set to zero: $2x + y = 0$ and $x + 4y = 0$. Solving this system gives the only critical point $(0, 0)$.
    - Compute second partials: $f_{xx} = 2$, $f_{yy} = 4$, $f_{xy} = 1$.
    - Calculate determinant $D = (2)(4) - (1)^2 = 7$.
    - Since $D > 0$ and $f_{xx} = 2 > 0$, the point is a local minimum.

    Answer: Local minimum at $(0, 0)$

2. Classify the critical point at $(-1, 0)$ for the function $f(x, y) = x^2 - 3y^2 + 2x$.

    - Verify critical point: $f_x = 2x + 2$, $f_y = -6y$. Setting these to zero gives $(-1, 0)$.
    - Compute second partials: $f_{xx} = 2$, $f_{yy} = -6$, $f_{xy} = 0$.
    - Calculate determinant $D = (2)(-6) - 0^2 = -12$.
    - Since $D < 0$, the point is a saddle point.

    Answer: Saddle point at $(-1, 0)$

3. Find and classify all critical points of $f(x, y) = x^3 - 3x + y^2$.

    - Compute partials: $f_x = 3x^2 - 3$ and $f_y = 2y$.
    - Set to zero: $3x^2 = 3 \implies x = \pm 1$ and $2y = 0 \implies y = 0$. Critical points are $(1, 0)$ and $(-1, 0)$.
    - Compute second partials: $f_{xx} = 6x$, $f_{yy} = 2$, $f_{xy} = 0$.
    - At $(1, 0)$: $D = (6)(2) - 0^2 = 12$. Since $D > 0$ and $f_{xx} = 6 > 0$, it is a local minimum.
    - At $(-1, 0)$: $D = (-6)(2) - 0^2 = -12$. Since $D < 0$, it is a saddle point.

    Answer: Local minimum at $(1, 0)$ and saddle point at $(-1, 0)$
