Taylor expansion in two variables; the Hessian

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Practice problems

1. Find the second-order Taylor expansion of $f(x, y) = \sin(x) + \cos(y)$ centered at $(0, 0)$.
2. Compute the Hessian matrix for $f(x, y) = x^3y^2$ at the point $(1, 2)$.
3. For $f(x, y) = e^{x+y}$, show that the second-order Taylor expansion at $(0, 0)$ is $1 + x + y + \frac{1}{2}(x^2 + 2xy + y^2)$.

Solutions

1. Find the second-order Taylor expansion of $f(x, y) = \sin(x) + \cos(y)$ centered at $(0, 0)$.

   - Compute $f(0, 0) = \sin(0) + \cos(0) = 1$.
   - Compute first partials: $f_x = \cos(x)$, $f_y = -\sin(y)$. At $(0, 0)$, $f_x = 1$ and $f_y = 0$.
   - Compute second partials: $f_{xx} = -\sin(x)$, $f_{yy} = -\cos(y)$, $f_{xy} = 0$. At $(0, 0)$, $f_{xx} = 0$, $f_{yy} = -1$, $f_{xy} = 0$.
   - Plug into formula: $f(x, y) \approx 1 + 1(x - 0) + 0(y - 0) + \frac{1}{2}(0x^2 + 2(0)xy + (-1)y^2)$.

   Answer: f(x, y) \approx 1 + x - \frac12y^2

2. Compute the Hessian matrix for $f(x, y) = x^3y^2$ at the point $(1, 2)$.

   - First partials: $f_x = 3x^2y^2$, $f_y = 2x^3y$.
   - Second partials: $f_{xx} = 6xy^2$, $f_{yy} = 2x^3$, $f_{xy} = 6x^2y$.
   - Evaluate at $(1, 2)$: $f_{xx} = 6(1)(4) = 24$, $f_{yy} = 2(1)^3 = 2$, $f_{xy} = 6(1)^2(2) = 12$.

   Answer: H = \beginpmatrix 24 & 12 \\ 12 & 2 \endpmatrix

3. For $f(x, y) = e^{x+y}$, show that the second-order Taylor expansion at $(0, 0)$ is $1 + x + y + \frac{1}{2}(x^2 + 2xy + y^2)$.

   - Value: $f(0, 0) = e^0 = 1$.
   - First partials: $f_x = e^{x+y}$, $f_y = e^{x+y}$. At $(0, 0)$, both are 1.
   - Second partials: $f_{xx} = e^{x+y}$, $f_{yy} = e^{x+y}$, $f_{xy} = e^{x+y}$. At $(0, 0)$, all are 1.
   - Assemble: $f(x, y) \approx 1 + 1x + 1y + \frac{1}{2}(1x^2 + 2(1)xy + 1y^2)$.

   Answer: f(x, y) \approx 1 + x + y + \frac12(x^2 + 2xy + y^2)
