The multivariable chain rule — tree diagrams and the matrix view

Lesson 14 · Module 1 — Differentiation in Several Variables · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

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Practice problems

1. Given $z = x^2 y$, where $x = e^t$ and $y = t^2$, find the total derivative $\frac{dz}{dt}$ at $t = 1$.
2. Let $f(x, y) = x + 2y$. Let $x = u^2 - v^2$ and $y = 2uv$. Use the Jacobian matrix to find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.
3. A function w depends on x, y, z, and each of these depends on u and v. Write the general expression for $\frac{\partial w}{\partial u}$ using summation notation.

Solutions

1. Given $z = x^2 y$, where $x = e^t$ and $y = t^2$, find the total derivative $\frac{dz}{dt}$ at $t = 1$.

   - Identify dependencies: z depends on x and y, and both depend on t.
   - Apply chain rule: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.
   - Compute partials: $\frac{\partial z}{\partial x} = 2xy$, $\frac{\partial z}{\partial y} = x^2$.
   - Compute inner derivatives: $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = 2t$.
   - Substitute: $\frac{dz}{dt} = (2xy)(e^t) + (x^2)(2t)$.
   - Evaluate at $t = 1$: $x = e$, $y = 1$. So $\frac{dz}{dt} = (2e)(e) + (e^2)(2) = 2e^2 + 2e^2 = 4e^2$.

   Answer: 4eˆ2

2. Let $f(x, y) = x + 2y$. Let $x = u^2 - v^2$ and $y = 2uv$. Use the Jacobian matrix to find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

   - Find the gradient of f: $\nabla f = \begin{pmatrix} 1 & 2 \end{pmatrix}$.
   - Find the Jacobian of the transformation: $J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$.
   - Multiply ∇f by J: $\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = \begin{pmatrix} 1(2u) + 2(2v) & 1(-2v) + 2(2u) \end{pmatrix}$.
   - Simplify: $\begin{pmatrix} 2u + 4v & 4u - 2v \end{pmatrix}$.

   Answer: \frac\partial f\partial u = 2u + 4v, \frac\partial f\partial v = 4u - 2v

3. A function w depends on x, y, z, and each of these depends on u and v. Write the general expression for $\frac{\partial w}{\partial u}$ using summation notation.

   - Identify the paths from w to u.
   - The paths are $w \to x \to u$, $w \to y \to u$, and $w \to z \to u$.
   - The contribution of each path is the product of the partial derivatives along that path.
   - Sum the contributions: $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$.
   - Express as a sum over indices $i = 1, 2, 3$ where $x_1 = x, x_2 = y, x_3 = z$.

   Answer: \frac\partial w\partial u = \sum_i=1ˆ3 \frac\partial w\partial x_i \frac\partial x_i\partial u

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