The gradient II — directional derivatives in full

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Practice problems

1. Find the directional derivative of $f(x, y) = x^2 + y^2$ at the point $(2, 3)$ in the direction of the vector $v = \langle 4, -3 \rangle$.
2. For the function $f(x, y) = 3x+2y$, find the directional derivative in the direction $\theta = \pi/4$ radians at any point (x, y).
3. Find the direction u (as a unit vector) in which the directional derivative of $f(x, y) = x^2y$ at $(1, 2)$ is exactly zero.

Solutions

1. Find the directional derivative of $f(x, y) = x^2 + y^2$ at the point $(2, 3)$ in the direction of the vector $v = \langle 4, -3 \rangle$.

   - Compute the gradient: $\nabla f = \langle 2x, 2y \rangle$.
   - Evaluate the gradient at $(2, 3)$: $\nabla f(2, 3) = \langle 4, 6 \rangle$.
   - Normalize the direction vector v: $|v| = \sqrt{4^2 + (-3)^2} = 5$, so $u = \langle 4/5, -3/5 \rangle$.
   - Compute the dot product: $D_u f = \langle 4, 6 \rangle \cdot \langle 4/5, -3/5 \rangle = 16/5 - 18/5 = -2/5$.

   Answer: -2/5

2. For the function $f(x, y) = 3x+2y$, find the directional derivative in the direction $\theta = \pi/4$ radians at any point (x, y).

   - Compute the gradient: $\nabla f = \langle 3, 2 \rangle$.
   - Find the unit vector for $\theta = \pi/4$: $u = \langle \cos(\pi/4), \sin(\pi/4) \rangle = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$.
   - Compute the dot product: $D_u f = \langle 3, 2 \rangle \cdot \langle \sqrt{2}/2, \sqrt{2}/2 \rangle = 3\sqrt{2}/2 + 2\sqrt{2}/2 = 5\sqrt{2}/2$.

   Answer: 5\sqrt2/2

3. Find the direction u (as a unit vector) in which the directional derivative of $f(x, y) = x^2 y$ at $(1, 2)$ is exactly zero.

   - Compute the gradient at $(1, 2)$: $\nabla f = \langle 2xy, x^2 \rangle \implies \nabla f(1, 2) = \langle 4, 1 \rangle$.
   - The directional derivative is zero when u is perpendicular to the gradient: $\langle 4, 1 \rangle \cdot \langle u_1, u_2 \rangle = 0$.
   - This implies $4u_1 + u_2 = 0$, or $u_2 = -4u_1$.
   - Since u is a unit vector, $u_1^2 + (-4u_1)^2 = 1 \implies 17u_1^2 = 1 \implies u_1 = \pm 1/\sqrt{17}$.
   - Thus, $u = \langle 1/\sqrt{17}, -4/\sqrt{17} \rangle$ or $u = \langle -1/\sqrt{17}, 4/\sqrt{17} \rangle$.

   Answer: \langle \pm 1/\sqrt17, \mp 4/\sqrt17 \rangle
