Functions of several variables; limits in 2D

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Practice problems

1. Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.
2. Determine if the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exists.
3. Prove whether $f(x,y) = \frac{x^3}{x^2+y^2}$ is continuous at $(0,0)$ if we define $f(0,0) = 0$.

Solutions

1. Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

   - Test the path $x = 0$: the function becomes $0/y^2 = 0$.
   - Test the path $y = 0$: the function becomes $0/x^2 = 0$.
   - Use the inequality $|x|/\sqrt{x^2+y^2} \leq 1$.
   - Rewrite the function as $|x| \cdot |y^2/(x^2+y^2)|$.
   - Since $y^2/(x^2+y^2) \leq 1$, the absolute value of the function is $\leq |x|$.
   - As $(x, y) \to (0, 0)$, $|x| \to 0$, so by the Squeeze Theorem, the limit is 0.

   Answer: 0

2. Determine if the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exists.

   - Test the path $y = 0$: the limit is $\lim_{x\to 0} x^2/x^2 = 1$.
   - Test the path $x = 0$: the limit is $\lim_{y\to 0} -y^2/y^2 = -1$.
   - Since $1 \neq -1$, the limit depends on the path.

   Answer: Does not exist

3. Prove whether $f(x, y) = \frac{x^3}{x^2+y^2}$ is continuous at $(0, 0)$ if we define $f(0, 0) = 0$.

   - Check the limit as $(x, y) \to (0, 0)$.
   - Convert to polar: $f(r, \theta) = (r^3 \cos^3(\theta))/(r^2) = r \cos^3(\theta)$.
   - Since $|\cos^3(\theta)| \leq 1$, we have $|f(r, \theta)| \leq r$.
   - As $r \to 0$, the limit is 0.
   - Since the limit 0 equals the defined value $f(0, 0) = 0$, the function is continuous.

   Answer: Continuous
