The TNB frame — a moving coordinate system on a curve

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Practice problems

1. Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ for the curve $\vec{r}(t) = \langle t, t^2, 0\rangle$ at $t = 1$.
2. For the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, verify that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$ for all t.
3. Find the unit binormal vector $\mathbf{B}(t)$ for the curve $\vec{r}(t) = \langle t, t^2, t^3\rangle$ at $t = 0$.

Solutions

1. Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ for the curve $\vec{r}(t) = \langle t, t^2, 0\rangle$ at $t = 1$.

   - Compute velocity: $\vec{v}(t) = \langle 1, 2t, 0\rangle$. At $t = 1$, $\vec{v}(1) = \langle 1, 2, 0\rangle$.
   - Normalize velocity: $|\vec{v}(1)| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$. So $\mathbf{T}(1) = \langle 1/\sqrt{5}, 2/\sqrt{5}, 0\rangle$.
   - Find $\mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t, 0\rangle$.
   - Differentiate $\mathbf{T}(t)$ using the quotient rule to find $\mathbf{T}'(t) = \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2, 0\rangle$.
   - At $t = 1$, $\mathbf{T}'(1) = \frac{1}{5\sqrt{5}}\langle -4, 2, 0\rangle$. Normalize this to find $\mathbf{N}(1) = \langle -4/\sqrt{20}, 2/\sqrt{20}, 0\rangle = \langle -2/\sqrt{5}, 1/\sqrt{5}, 0\rangle$.

   Answer: \mathbfT(1) = \langle \frac1\sqrt5, \frac2\sqrt5, 0 \rangle, \mathbfN(1) = \langle \frac-2\sqrt5, \frac1\sqrt5, 0 \rangle

2. For the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, verify that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$ for all t.

   - From the lesson, $\mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$.
   - From the lesson, $\mathbf{N}(t) = \langle -\cos t, -\sin t, 0\rangle$.
   - Compute the dot product: $\mathbf{T} \cdot \mathbf{N} = \frac{1}{\sqrt{2}}((-\sin t)(-\cos t) + (\cos t)(-\sin t) + (1)(0))$.
   - Simplify: $\frac{1}{\sqrt{2}}(\sin t \cos t - \sin t \cos t + 0) = 0$.

   Answer: The dot product is 0, confirming they are orthogonal.

3. Find the unit binormal vector $\mathbf{B}(t)$ for the curve $\vec{r}(t) = \langle t, t^2, t^3\rangle$ at $t = 0$.

   - Velocity: $\vec{v}(t) = \langle 1, 2t, 3t^2\rangle$. At $t = 0$, $\vec{v}(0) = \langle 1, 0, 0\rangle$.
   - Unit tangent: $\mathbf{T}(0) = \langle 1, 0, 0\rangle$.
   - Find $\mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2+9t^4}}\langle 1, 2t, 3t^2\rangle$.
   - Differentiate $\mathbf{T}(t)$ and evaluate at $t = 0$. $\mathbf{T}'(0) = \langle 0, 2, 0\rangle$.
   - Normalize to get $\mathbf{N}(0) = \langle 0, 1, 0\rangle$.
   - Compute $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \langle 0, 0, 1\rangle$.

   Answer: \mathbfB(0) = \langle 0, 0, 1 \rangle
