Arc length and curvature — how bent is a path?

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Practice problems

1. Find the arc length of the curve $\vec{r}(t) = \langle t, \frac{2}{3}t^{3/2}, 0\rangle$ from $t = 0$ to $t = 3$.
2. Calculate the curvature κ of the curve $\vec{r}(t) = \langle t, t^2, 0\rangle$ at the point $t = 0$.
3. A curve is given by $\vec{r}(t) = \langle e^t, e^t, t\rangle$. Find the curvature κ as a function of t.

Solutions

1. Find the arc length of the curve $\vec{r}(t) = \langle t, \frac{2}{3}t^{3/2}, 0\rangle$ from $t = 0$ to $t = 3$.

    - Compute the velocity vector: $\vec{v}(t) = \langle 1, t^{1/2}, 0\rangle$.
    - Compute the speed: $|\vec{v}(t)| = \sqrt{1^2 + (t^{1/2})^2 + 0^2} = \sqrt{1+t}$.
    - Set up the integral: $L = \int_0^3 \sqrt{1+t}\, dt$.
    - Use u-substitution with $u = 1 + t$, $du = dt$: $\int_1^4 u^{1/2}\, du$.
    - Evaluate: $[\frac{2}{3}u^{3/2}]_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8-1) = \frac{14}{3}$.

    Answer: 14/3

2. Calculate the curvature κ of the curve $\vec{r}(t) = \langle t, t^2, 0\rangle$ at the point $t = 0$.

    - Compute velocity: $\vec{v}(t) = \langle 1, 2t, 0\rangle$. At $t = 0$, $\vec{v}(0) = \langle 1, 0, 0\rangle$.
    - Compute acceleration: $\vec{a}(t) = \langle 0, 2, 0\rangle$. At $t = 0$, $\vec{a}(0) = \langle 0, 2, 0\rangle$.
    - Compute cross product: $\vec{v}(0) \times \vec{a}(0) = \langle 1, 0, 0\rangle \times \langle 0, 2, 0\rangle = \langle 0, 0, 2\rangle$.
    - Find magnitudes: $|\vec{v} \times \vec{a}| = 2$ and $|\vec{v}| = 1$.
    - Apply formula: $\kappa = \frac{2}{1^3} = 2$.

    Answer: 2

3. A curve is given by $\vec{r}(t) = \langle e^t, e^t, t\rangle$. Find the curvature κ as a function of t.

    - Compute velocity: $\vec{v}(t) = \langle e^t, e^t, 1\rangle$.
    - Compute acceleration: $\vec{a}(t) = \langle e^t, e^t, 0\rangle$.
    - Compute cross product: $\vec{v} \times \vec{a} = \langle -e^t, e^t, 0\rangle$.
    - Find magnitudes: $|\vec{v} \times \vec{a}| = \sqrt{(-e^t)^2 + (e^t)^2} = \sqrt{2e^{2t}} = e^t\sqrt{2}$.
    - Find speed: $|\vec{v}| = \sqrt{(e^t)^2 + (e^t)^2 + 1^2} = \sqrt{2e^{2t}+1}$.
    - Apply formula: $\kappa(t) = \frac{e^t\sqrt{2}}{(2e^{2t}+1)^{3/2}}$.

    Answer: \frace^t\sqrt2(2e^2t + 1)^3/2
