

# Module 6 Review and Self-Test

Lesson 88 · Module 6 — Series Solutions & Special Functions · Differential Equations · daily-maths.uk/diffeq

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## Practice problems

1. Find the indicial equation and the roots  $r_1, r_2$  for the differential equation  $2x^2y'' + xy' - (x+1)y = 0$  at the singular point  $x = 0$ .
2. Determine if the function  $y(x) = J_1(x)$  is a solution to the equation  $x^2y'' + xy' + (x^2 - 1)y = 0$ . Justify using the general form of the Bessel equation.
3. Given a function  $f(x)$  on  $[-1, 1]$ , write the integral formula for the coefficient  $c_2$  in the Legendre expansion  $f(x) = \sum c_n P_n(x)$ , and state the value of the norm squared  $\int_{-1}^1 [P_2(x)]^2 dx$ .

## Solutions

1. Find the indicial equation and the roots  $r_1, r_2$  for the differential equation  $2x^2y'' + xy' - (x+1)y = 0$  at the singular point  $x = 0$ .

- Divide by  $2x^2$  to get  $y'' + \frac{1}{2x}y' - \frac{x+1}{2x^2}y = 0$ .
- Identify  $P(x) = \frac{1}{2x}$  and  $Q(x) = -\frac{x+1}{2x^2}$ .
- Compute  $p_0 = \lim_{x \rightarrow 0} xP(x) = \frac{1}{2}$  and  $q_0 = \lim_{x \rightarrow 0} x^2Q(x) = -\frac{1}{2}$ .
- The indicial equation is  $r(r-1) + \frac{1}{2}r - \frac{1}{2} = 0$ .
- Simplify to  $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$ , or  $2r^2 - r - 1 = 0$ .
- Factor as  $(2r+1)(r-1) = 0$ , giving  $r_1 = 1$  and  $r_2 = -1/2$ .

**Answer:**  $r_1 = 1, r_2 = -1/2$

2. Determine if the function  $y(x) = J_1(x)$  is a solution to the equation  $x^2y'' + xy' + (x^2 - 1)y = 0$ . Justify using the general form of the Bessel equation.

- The general Bessel equation is  $x^2y'' + xy' + (x^2 - n^2)y = 0$ .
- Compare the given equation  $x^2y'' + xy' + (x^2 - 1)y = 0$  to the general form.
- We see that  $n^2 = 1$ , which implies  $n = 1$  or  $n = -1$ .
- By definition,  $J_n(x)$  is a solution to the Bessel equation of order  $n$ .
- Since  $n = 1$  satisfies the equation,  $J_1(x)$  is indeed a solution.

**Answer:** Yes, it is a solution because it matches the Bessel equation of order  $n=1$ .

3. Given a function  $f(x)$  on  $[-1, 1]$ , write the integral formula for the coefficient  $c_2$  in the Legendre expansion  $f(x) = \sum c_n P_n(x)$ , and state the value of the norm squared  $\int_{-1}^1 [P_2(x)]^2 dx$ .

- The coefficient formula is  $c_n = \frac{\int_{-1}^1 f(x) P_n(x) dx}{\int_{-1}^1 [P_n(x)]^2 dx}$ .
- For  $n = 2$ ,  $c_2 = \frac{\int_{-1}^1 f(x) P_2(x) dx}{\int_{-1}^1 [P_2(x)]^2 dx}$ .
- The general formula for the norm of Legendre polynomials is  $\int_{-1}^1 [P_n(x)]^2 dx = \frac{2}{2n+1}$ .
- Substitute  $n = 2$  into the norm formula:  $\frac{2}{2(2)+1} = \frac{2}{5}$ .

**Answer:**  $c_2 = \frac{5}{2} \int_{-1}^1 f(x) P_2(x) dx$ ;  $\text{norm squared} = 2/5$