

Special functions in physics — where they all live

Lesson 87 · Module 6 — Series Solutions & Special Functions · Differential Equations · daily-maths.uk/diffeq

Practice problems

1. A vibrating circular membrane of radius $R = 1$ is fixed at the edges. Which special function describes the radial displacement $u(r)$, and what is the condition on the order $n = 0$ for the first mode of vibration?
2. A physical quantity V on the surface of a sphere depends only on the colatitude θ . If the solution is a polynomial of degree $n = 2$, write the form of the solution using the known Legendre polynomials.
3. Explain why the Gamma function is necessary to define the Bessel function $J_n(x)$ for a non-integer order, such as $n = 1/2$.

Solutions

1. A vibrating circular membrane of radius $R = 1$ is fixed at the edges. Which special function describes the radial displacement $u(r)$, and what is the condition on the order $n = 0$ for the first mode of vibration?
 - The system has cylindrical symmetry, so the radial part satisfies Bessel's equation of order zero.
 - The solution is $u(r) = AJ_0(kr)$.
 - The boundary condition is $u(1) = 0$, which means $J_0(k) = 0$.
 - For the first mode, k must be the first positive zero of J_0 .

Answer: Bessel function $J_0(kr)$ with $J_0(k) = 0$.

2. A physical quantity V on the surface of a sphere depends only on the colatitude θ . If the solution is a polynomial of degree $n = 2$, write the form of the solution using the known Legendre polynomials.
 - Spherical symmetry with angular dependence leads to Legendre's equation.
 - The solutions are Legendre polynomials $P_n(x)$ where $x = \cos(\theta)$.
 - For $n = 2$, the solution is $V(\theta) = CP_2(\cos(\theta))$.
 - Using the formula $P_2(x) = \frac{1}{2}(3x^2 - 1)$, we have $V(\theta) = \frac{C}{2}(3\cos^2(\theta) - 1)$.

Answer: $V(\theta) = CP_2(\cos(\theta))$

3. Explain why the Gamma function is necessary to define the Bessel function $J_n(x)$ for a non-integer order, such as $n = 1/2$.
 - The power series for $J_n(x)$ contains a factorial term $(k + n)!$ in the denominator.
 - Factorials are only defined for non-negative integers.
 - The Gamma function extends the factorial such that $\Gamma(k + n + 1)$ is defined for any real $k + n$.
 - This allows the series to converge and define a valid function for fractional orders.

Answer: It generalizes the factorial in the series denominator to non-integer values.