

The Gamma Function Returns

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Practice problems

1. Given the recurrence relation $a_k = \frac{a_{k-1}}{k+2}$ with $a_0 = 1$, find a closed form expression for a_k using the Gamma function.
2. Use the general formula for $J_n(x)$ to write the first two terms of the series for $J_{3/2}(x)$.
3. Show that for integer n , the general formula $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} (x/2)^{n+2k}$ simplifies to the form using $(n+k)!$ in the denominator.

Solutions

1. Given the recurrence relation $a_k = \frac{a_{k-1}}{k+2}$ with $a_0 = 1$, find a closed form expression for a_k using the Gamma function.

- Write out the first few terms: $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3 \cdot 4}$, $a_3 = \frac{1}{3 \cdot 4 \cdot 5}$.
- Observe the pattern: $a_k = \frac{1}{3 \cdot 4 \cdots (k+2)}$.
- Multiply top and bottom by $2!$ to complete the factorial: $a_k = \frac{2}{(k+2)!}$.
- Replace the factorial with the Gamma function: $a_k = \frac{2}{\Gamma(k+3)}$.

Answer: $a_k = \frac{2}{\Gamma(k+3)}$

2. Use the general formula for $J_n(x)$ to write the first two terms of the series for $J_{3/2}(x)$.

- For $k = 0$, the term is $\frac{(-1)^0}{0! \Gamma(3/2+1)} (x/2)^{3/2} = \frac{1}{\Gamma(5/2)} (x/2)^{3/2}$.
- Calculate $\Gamma(5/2) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$.
- The first term is $\frac{4}{3\sqrt{\pi}} (x/2)^{3/2}$.
- For $k = 1$, the term is $\frac{(-1)^1}{1! \Gamma(3/2+2)} (x/2)^{3/2+2} = \frac{-1}{\Gamma(7/2)} (x/2)^{7/2}$.
- Calculate $\Gamma(7/2) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$.
- The second term is $-\frac{8}{15\sqrt{\pi}} (x/2)^{7/2}$.

Answer: $J_{3/2}(x) \approx \frac{4}{3\sqrt{\pi}} (x/2)^{3/2} - \frac{8}{15\sqrt{\pi}} (x/2)^{7/2}$

3. Show that for integer n , the general formula $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} (x/2)^{n+2k}$ simplifies to the form using $(n+k)!$ in the denominator.

- Identify the term $\Gamma(n+k+1)$ in the denominator.
- Since n and k are non-negative integers, $n+k$ is also a non-negative integer.
- Apply the identity $\Gamma(m+1) = m!$ where $m = n+k$.
- Substitute $\Gamma(n+k+1) = (n+k)!$ into the summation.
- The formula becomes $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} (x/2)^{n+2k}$.

Answer: $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} \left(\frac{x}{2}\right)^{n+2k}$