

Legendre's equation and Legendre polynomials

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Practice problems

1. Verify that $P_2(x) = \frac{1}{2}(3x^2 - 1)$ satisfies Legendre's equation for $n = 2$.
2. Using the recurrence relation $a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)(k+2)}a_k$, find the ratio a_2/a_0 for the case $n = 4$.
3. Determine the value of a_1 for $P_1(x)$ given the normalization $P_1(1) = 1$.

Solutions

1. Verify that $P_2(x) = \frac{1}{2}(3x^2 - 1)$ satisfies Legendre's equation for $n = 2$.

- Compute $P_2'(x) = 3x$ and $P_2''(x) = 3$.
- Substitute into $(1 - x^2)y'' - 2xy' + 6y$.
- $(1 - x^2)(3) - 2x(3x) + 6(\frac{3}{2}x^2 - \frac{1}{2}) = 3 - 3x^2 - 6x^2 + 9x^2 - 3$.
- The terms cancel to 0.

Answer: The equation is satisfied as the result is 0.

2. Using the recurrence relation $a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)(k+2)}a_k$, find the ratio a_2/a_0 for the case $n = 4$.

- Set $n = 4$ and $k = 0$ in the recurrence relation.
- $a_2 = \frac{0(1)-4(5)}{(1)(2)}a_0$.
- $a_2 = \frac{-20}{2}a_0 = -10a_0$.
- The ratio a_2/a_0 is -10 .

Answer: -10

3. Determine the value of a_1 for $P_1(x)$ given the normalization $P_1(1) = 1$.

- For $n = 1$, the recurrence relation gives $a_3 = \frac{1(2)-1(2)}{6}a_1 = 0$.
- Thus $P_1(x) = a_0 + a_1x$. However, for $n = 1$, the even series a_0 must be zero for the solution to be a polynomial of degree 1.
- So $P_1(x) = a_1x$.
- Apply normalization: $P_1(1) = a_1(1) = 1$.
- Therefore $a_1 = 1$.

Answer: 1