

Bessel's Equation and Bessel Functions

Lesson 83 · Module 6 — Series Solutions & Special Functions · Differential Equations · daily-maths.uk/diffeq

Practice problems

1. Identify the order n of the differential equation $x^2y'' + xy' + (x^2 - 9)y = 0$ and state the general solution in terms of Bessel functions.
2. A circular membrane of radius $R = 1$ is vibrating in its fundamental mode (order $n = 0$). Given that the first zero of $J_0(x)$ is approximately 2.4048, what is the value of the scaling factor k if the radial solution is $y(r) = J_0(kr)$ and the boundary condition is $y(1) = 0$?
3. Determine if the equation $y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0$ is a Bessel equation. If so, find its order n .

Solutions

1. Identify the order n of the differential equation $x^2y'' + xy' + (x^2 - 9)y = 0$ and state the general solution in terms of Bessel functions.
 - Compare the equation to the standard form $x^2y'' + xy' + (x^2 - n^2)y = 0$.
 - Observe that $n^2 = 9$, which means $n = 3$.
 - The general solution is a linear combination of the first and second kind functions of order 3.

Answer: $n = 3$, $y(x) = C_1 J_3(x) + C_2 Y_3(x)$

2. A circular membrane of radius $R = 1$ is vibrating in its fundamental mode (order $n = 0$). Given that the first zero of $J_0(x)$ is approximately 2.4048, what is the value of the scaling factor k if the radial solution is $y(r) = J_0(kr)$ and the boundary condition is $y(1) = 0$?
 - The boundary condition requires $J_0(k \cdot 1) = 0$.
 - This means k must be a zero of the Bessel function J_0 .
 - For the fundamental mode, we use the first zero.

Answer: $k \approx 2.4048$

3. Determine if the equation $y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0$ is a Bessel equation. If so, find its order n .
 - Multiply the entire equation by x^2 to bring it to standard form.
 - The equation becomes $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$.
 - Compare this to $x^2y'' + xy' + (x^2 - n^2)y = 0$.
 - We find $n^2 = 1/4$, so $n = 1/2$.

Answer: Yes, it is a Bessel equation of order $n = 1/2$