

The Method of Frobenius

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Practice problems

1. For the equation $2x^2y'' + 3xy' + (x^2 - 1)y = 0$, determine if $x = 0$ is a regular singular point and find the roots of the indicial equation.
2. Find the first three non-zero terms of the Frobenius solution for $xy'' + 2y' + xy = 0$ corresponding to the larger root of the indicial equation.
3. Explain why the standard power series method fails for the equation $x^2y'' + y = 0$ at $x = 0$, and find the Frobenius exponents.

Solutions

1. For the equation $2x^2y'' + 3xy' + (x^2 - 1)y = 0$, determine if $x = 0$ is a regular singular point and find the roots of the indicial equation.

- Divide by $2x^2$ to get $y'' + \frac{3}{2x}y' + \frac{x^2-1}{2x^2}y = 0$.
- Compute $p_0 = \lim_{x \rightarrow 0} x(\frac{3}{2x}) = 1.5$.
- Compute $q_0 = \lim_{x \rightarrow 0} x^2(\frac{x^2-1}{2x^2}) = -0.5$.
- Set up the indicial equation: $r(r-1) + 1.5r - 0.5 = 0$.
- Simplify to $r^2 + 0.5r - 0.5 = 0$.
- Solve using the quadratic formula: $r = \frac{-0.5 \pm \sqrt{0.25+2}}{2} = \frac{-0.5 \pm 1.5}{2}$.

Answer: $x=0$ is a regular singular point; roots are $r_1 = 0.5, r_2 = -1$.

2. Find the first three non-zero terms of the Frobenius solution for $xy'' + 2y' + xy = 0$ corresponding to the larger root of the indicial equation.

- Normalize: $y'' + \frac{2}{x}y' + y = 0$, so $p_0 = 2, q_0 = 0$.
- Indicial equation: $r(r-1) + 2r + 0 = 0 \implies r^2 + r = 0$. Roots are $r = 0, r = -1$.
- Use $r = 0$. The equation is $xy'' + 2y' + xy = 0$.
- Substitute $y = \sum a_k x^k$: $\sum k(k-1)a_k x^{k-1} + \sum 2ka_k x^{k-1} + \sum a_k x^{k+1} = 0$.
- Combine first two: $\sum k(k+1)a_k x^{k-1}$.
- Recurrence: $(k+2)(k+3)a_{k+2} = -a_k$.
- For $a_0 = 1$: $a_1 = 0, a_2 = -1/(2 \cdot 3) = -1/6, a_3 = 0, a_4 = -a_2/(4 \cdot 5) = 1/120$.

Answer: $y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \dots$

3. Explain why the standard power series method fails for the equation $x^2y'' + y = 0$ at $x = 0$, and find the Frobenius exponents.

- Normalize: $y'' + \frac{1}{x^2}y = 0$.
- Check $p_0 = \lim x(0) = 0$ (finite).
- Check $q_0 = \lim x^2(\frac{1}{x^2}) = 1$ (finite).
- It is a regular singular point, but the solution is not a polynomial because the indicial equation is $r(r-1) + 0r + 1 = 0$.
- Simplify: $r^2 - r + 1 = 0$.
- Solve: $r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

Answer: Standard power series fail because the exponents are complex; the Frobenius exponents are $r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.