

Ordinary vs singular points; Cauchy-Euler equations

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Practice problems

1. Determine if $t = 0$ is an ordinary point or a singular point for the equation $ty'' + 2y' + ty = 0$. If it is singular, determine if it is a regular singular point.
2. Solve the Cauchy-Euler equation $t^2y'' - 2ty' + 2y = 0$ for $t > 0$.
3. Solve the Cauchy-Euler equation $t^2y'' + 3ty' + 2.25y = 0$ for $t > 0$.

Solutions

1. Determine if $t = 0$ is an ordinary point or a singular point for the equation $ty'' + 2y' + ty = 0$. If it is singular, determine if it is a regular singular point.

- Divide by t to get standard form: $y'' + \frac{2}{t}y' + y = 0$.
- The coefficient $p(t) = \frac{2}{t}$ is discontinuous at $t = 0$, so $t = 0$ is a singular point.
- Check for regularity: $\lim_{t \rightarrow 0} tp(t) = \lim_{t \rightarrow 0} t(\frac{2}{t}) = 2$.
- Check the second coefficient: $\lim_{t \rightarrow 0} t^2q(t) = \lim_{t \rightarrow 0} t^2(1) = 0$.
- Since both limits are finite, $t = 0$ is a regular singular point.

Answer: Regular singular point

2. Solve the Cauchy-Euler equation $t^2y'' - 2ty' + 2y = 0$ for $t > 0$.

- Assume $y = t^r$, leading to the indicial equation $r(r - 1) - 2r + 2 = 0$.
- Simplify: $r^2 - r - 2r + 2 = r^2 - 3r + 2 = 0$.
- Factor: $(r - 1)(r - 2) = 0$, so $r_1 = 1$ and $r_2 = 2$.
- The general solution is $y(t) = C_1t^1 + C_2t^2$.

Answer: $y(t) = C_1t + C_2t^2$

3. Solve the Cauchy-Euler equation $t^2y'' + 3ty' + 2.25y = 0$ for $t > 0$.

- Assume $y = t^r$, leading to the indicial equation $r(r - 1) + 3r + 2.25 = 0$.
- Simplify: $r^2 + 2r + 2.25 = 0$.
- Use the quadratic formula: $r = \frac{-2 \pm \sqrt{4 - 4(2.25)}}{2} = \frac{-2 \pm \sqrt{4 - 9}}{2} = \frac{-2 \pm \sqrt{-5}}{2}$.
- The roots are $r = -1 \pm i\frac{\sqrt{5}}{2}$.
- Using the complex root formula with $\alpha = -1$ and $\beta = \frac{\sqrt{5}}{2}$, the solution is $y(t) = t^{-1}[C_1 \cos(\frac{\sqrt{5}}{2} \ln t) + C_2 \sin(\frac{\sqrt{5}}{2} \ln t)]$.

Answer: $y(t) = t^{-1}[C_1 \cos(\frac{\sqrt{5}}{2} \ln t) + C_2 \sin(\frac{\sqrt{5}}{2} \ln t)]$