

Power series solutions I — the method

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Practice problems

1. Find the first four non-zero coefficients of the power series solution for $y'' - y = 0$ given the initial conditions $y(0) = 1$ and $y'(0) = 1$.
2. For the equation $y'' + 4y = 0$, derive the recurrence relation for the coefficients a_k .
3. Using the recurrence relation $a_{k+2} = \frac{-a_k}{(k+2)(k+1)}$ and initial conditions $y(0) = 1, y'(0) = 0$, find the first three non-zero terms of the series solution.

Solutions

1. Find the first four non-zero coefficients of the power series solution for $y'' - y = 0$ given the initial conditions $y(0) = 1$ and $y'(0) = 1$.

- Assume $y = \sum a_k x^k$, then $y'' = \sum k(k-1)a_k x^{k-2}$.
- Substitute into $y'' - y = 0$: $\sum (k+2)(k+1)a_{k+2}x^k - \sum a_k x^k = 0$.
- Recurrence relation: $a_{k+2} = a_k / ((k+2)(k+1))$.
- Given $a_0 = 1, a_1 = 1$.
- Compute $a_2 = a_0 / (2 \cdot 1) = 1/2$.
- Compute $a_3 = a_1 / (3 \cdot 2) = 1/6$.

Answer: $a_0 = 1, a_1 = 1, a_2 = 1/2, a_3 = 1/6$

2. For the equation $y'' + 4y = 0$, derive the recurrence relation for the coefficients a_k .

- Substitute $y = \sum a_k x^k$ and $y'' = \sum (k+2)(k+1)a_{k+2}x^k$.
- The equation becomes $\sum [(k+2)(k+1)a_{k+2} + 4a_k]x^k = 0$.
- Set the coefficient to zero: $(k+2)(k+1)a_{k+2} + 4a_k = 0$.

Answer: $a_{k+2} = \frac{-4a_k}{(k+2)(k+1)}$

3. Using the recurrence relation $a_{k+2} = \frac{-a_k}{(k+2)(k+1)}$ and initial conditions $y(0) = 1, y'(0) = 0$, find the first three non-zero terms of the series solution.

- Initial conditions give $a_0 = 1$ and $a_1 = 0$.
- Since $a_1 = 0$, all odd coefficients $a_3, a_5, \dots$ are zero.
- Compute $a_2 = -a_0 / (2 \cdot 1) = -1/2$.
- Compute $a_4 = -a_2 / (4 \cdot 3) = -(-1/2) / 12 = 1/24$.
- The terms are $a_0 x^0, a_2 x^2$, and $a_4 x^4$.

Answer: $y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$