

Module 5 review + self-test

Lesson 78 · Module 5 — Nonlinear Dynamics & Chaos · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Find the equilibrium points of the system $x' = 2x - xy$ and $y' = xy - 3y$, and determine the stability of each using the Jacobian matrix.
2. For the logistic map $x_{n+1} = rx_n(1 - x_n)$, find the non-zero fixed point x^* in terms of r and determine the value of r at which this fixed point becomes unstable.
3. A chaotic system has a Lyapunov exponent of $\lambda = 0.12$. If the initial uncertainty in the state is $\epsilon_0 = 10^{-6}$ and the maximum tolerable error is $\epsilon_{max} = 0.1$, calculate the horizon of predictability $t_{horizon}$.

Solutions

1. Find the equilibrium points of the system $x' = 2x - xy$ and $y' = xy - 3y$, and determine the stability of each using the Jacobian matrix.

- Set $2x - xy = 0$ and $xy - 3y = 0$.
- From the second equation, $y(x - 3) = 0$, so $y = 0$ or $x = 3$.
- If $y = 0$, then $2x = 0$, so $(0, 0)$ is an equilibrium.
- If $x = 3$, then $6 - 3y = 0$, so $y = 2$, giving $(3, 2)$ as an equilibrium.
- Compute Jacobian $J = \begin{pmatrix} 2 - y & -x \\ y & x - 3 \end{pmatrix}$.
- At $(0, 0)$, $J = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, eigenvalues $2, -3$ (Saddle).
- At $(3, 2)$, $J = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$, eigenvalues $\pm i\sqrt{6}$ (Center/Spiral).

Answer: Equilibria at $(0, 0)$ (Saddle) and $(3, 2)$ (Center/Spiral).

2. For the logistic map $x_{n+1} = rx_n(1 - x_n)$, find the non-zero fixed point x^* in terms of r and determine the value of r at which this fixed point becomes unstable.

- Set $x = rx(1 - x)$.
- Divide by x (assuming $x \neq 0$): $1 = r(1 - x)$.
- Solve for x : $1/r = 1 - x \implies x^* = 1 - 1/r$.
- Stability requires $|f'(x^*)| < 1$.
- Compute $f'(x) = r - 2rx$.
- Evaluate at x^* : $f'(1 - 1/r) = r - 2r(1 - 1/r) = r - 2r + 2 = 2 - r$.
- Instability occurs when $|2 - r| > 1$, which means $r > 3$ (since r is positive).

Answer: Fixed point $x^* = 1 - 1/r$; becomes unstable at $r = 3$.

3. A chaotic system has a Lyapunov exponent of $\lambda = 0.12$. If the initial uncertainty in the state is $\epsilon_0 = 10^{-6}$ and the maximum tolerable error is $\epsilon_{max} = 0.1$, calculate the horizon of predictability $t_{horizon}$.

- Use the formula $\epsilon_{max} = \epsilon_0 e^{\lambda t}$.
- Substitute values: $0.1 = 10^{-6} e^{0.12t}$.
- Divide by 10^{-6} : $10^5 = e^{0.12t}$.
- Take natural log: $\ln(10^5) = 0.12t$.
- Calculate $\ln(10^5) \approx 11.513$.
- Solve for t : $t = 11.513/0.12 \approx 95.94$.

Answer: $t_{horizon} \approx 95.94$ time units.