

The logistic map — chaos in one line

Lesson 77 · Module 5 — Nonlinear Dynamics & Chaos · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Given the logistic map $x_{n+1} = 2.5x_n(1 - x_n)$ with $x_0 = 0.2$, compute the values of x_1, x_2, x_3 , and x_4 . Does the sequence appear to be converging to a fixed point?
2. For the logistic map $x_{n+1} = rx_n(1 - x_n)$, find the non-zero fixed point x^* in terms of r . For what value of r does this fixed point equal 0.75?
3. Explain why the logistic map can exhibit chaos while the continuous logistic differential equation $y' = ry(1 - y)$ cannot. Mention the role of dimensionality and time.

Solutions

1. Given the logistic map $x_{n+1} = 2.5x_n(1 - x_n)$ with $x_0 = 0.2$, compute the values of x_1, x_2, x_3 , and x_4 . Does the sequence appear to be converging to a fixed point?

- Step 1: $x_1 = 2.5(0.2)(1 - 0.2) = 2.5(0.2)(0.8) = 0.4$.
- Step 2: $x_2 = 2.5(0.4)(1 - 0.4) = 2.5(0.4)(0.6) = 0.6$.
- Step 3: $x_3 = 2.5(0.6)(1 - 0.6) = 2.5(0.6)(0.4) = 0.6$.
- Step 4: $x_4 = 2.5(0.6)(1 - 0.6) = 0.6$.
- The sequence converges to the fixed point 0.6.

Answer: $x_1=0.4$, $x_2=0.6$, $x_3=0.6$, $x_4=0.6$; yes, it converges to 0.6.

2. For the logistic map $x_{n+1} = rx_n(1 - x_n)$, find the non-zero fixed point x^* in terms of r . For what value of r does this fixed point equal 0.75?

- Step 1: Set $x_{n+1} = x_n = x^*$.
- Step 2: $x^* = rx^*(1 - x^*)$.
- Step 3: Divide by x^* (assuming $x^* \neq 0$): $1 = r(1 - x^*)$.
- Step 4: Solve for x^* : $1/r = 1 - x^* \implies x^* = 1 - 1/r$.
- Step 5: Set $0.75 = 1 - 1/r \implies 1/r = 0.25 \implies r = 4$.

Answer: $x^* = 1 - 1/r$; $r = 4$.

3. Explain why the logistic map can exhibit chaos while the continuous logistic differential equation $y' = ry(1 - y)$ cannot. Mention the role of dimensionality and time.

- Step 1: The continuous logistic equation is a first-order ODE in one dimension. In 1D continuous systems, trajectories cannot cross and must move monotonically toward an equilibrium or infinity, preventing oscillation or chaos.
- Step 2: The logistic map is a discrete-time system. This allows the state to 'jump' over the equilibrium point, creating oscillations.
- Step 3: For continuous systems to be chaotic, the Poincare-Bendixson theorem implies they need at least three dimensions (like the Lorenz system).
- Step 4: Therefore, the discrete nature of the map effectively adds the complexity needed for chaos in a single variable.

Answer: Continuous 1D systems are monotonic and cannot oscillate; discrete maps can jump over equilibria, allowing for period doubling and chaos.