The Lorenz system II — sensitive dependence

Practice problems

1. A chaotic system has a Lyapunov exponent of $\lambda = 0.5$. If the initial uncertainty in the state is $\delta(0) = 10^{-6}$, how long does it take for the uncertainty to grow to $\delta(t) = 1$? Give your answer to two decimal places.

2. Suppose you increase the precision of your initial measurement by a factor of 100 (reducing $\delta(0)$ by 100). For a system with $\lambda = 0.8$, by how much additional time Δt is the predictability horizon extended?

3. Explain why a system with a negative Lyapunov exponent cannot be chaotic, and relate this to the concept of an attractor.

Solutions

1. A chaotic system has a Lyapunov exponent of $\lambda = 0.5$. If the initial uncertainty in the state is $\delta(0) = 10^{-6}$, how long does it take for the uncertainty to grow to $\delta(t) = 1$? Give your answer to two decimal places.

   - Use the formula $\delta(t) = \delta(0)e^{\lambda t}$.
   - Substitute the values: $1 = 10^{-6}e^{0.5t}$.
   - Divide by 10^{-6}: $10^6 = e^{0.5t}$.
   - Take the natural logarithm of both sides: $\ln(10^6) = 0.5t$.
   - Solve for t: $t = \frac{6\ln(10)}{0.5} = 12\ln(10) \approx 12 \times 2.3025$.
   - Calculate the final value: $t \approx 27.63$.

   Answer: 27.63

2. Suppose you increase the precision of your initial measurement by a factor of 100 (reducing $\delta(0)$ by 100). For a system with $\lambda = 0.8$, by how much additional time Δt is the predictability horizon extended?

   - The time to reach a certain error ϵ is $t = \frac{1}{\lambda}\ln(\epsilon/\delta(0))$.
   - The extension is $\Delta t = t_{\text{new}} - t_{\text{old}} = \frac{1}{\lambda}\ln(\epsilon/(\delta(0)/100)) - \frac{1}{\lambda}\ln(\epsilon/\delta(0))$.
   - Simplify using log laws: $\Delta t = \frac{1}{\lambda}\ln(100)$.
   - Substitute $\lambda = 0.8$: $\Delta t = \frac{\ln(100)}{0.8}$.
   - Calculate: $\Delta t \approx \frac{4.605}{0.8} \approx 5.76$.

   Answer: 5.76

3. Explain why a system with a negative Lyapunov exponent cannot be chaotic, and relate this to the concept of an attractor.

   - A negative Lyapunov exponent $\lambda < 0$ implies that $\delta(t) = \delta(0)e^{\lambda t}$ decays toward zero as $t \to \infty$.
   - This means nearby trajectories converge rather than diverge, which is the opposite of sensitive dependence.
   - In terms of attractors, this indicates that the system is attracted to a stable fixed point or a stable limit cycle.
   - Since chaos requires the trajectories to diverge while remaining bounded (stretching and folding), a negative λ precludes the possibility of chaos.

   Answer: Nearby trajectories converge exponentially, meaning the system is stable and lacks the sensitive dependence required for chaos.