

The Lorenz system I — the butterfly discovered

Lesson 75 · Module 5 — Nonlinear Dynamics & Chaos · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. For a Lorenz system with $\sigma = 10$, $\beta = 8/3$, and $\rho = 15$, verify that the origin $(0, 0, 0)$ is an equilibrium point.
2. Find the non-zero equilibrium points for the Lorenz system when $\sigma = 10$, $\beta = 2$, and $\rho = 3$.
3. Explain why the Poincare-Bendixson theorem implies that the Lorenz system must have at least three variables to exhibit chaos.

Solutions

1. For a Lorenz system with $\sigma = 10$, $\beta = 8/3$, and $\rho = 15$, verify that the origin $(0, 0, 0)$ is an equilibrium point.

- Substitute $x = 0, y = 0, z = 0$ into the first equation: $x' = 10(0 - 0) = 0$.
- Substitute into the second equation: $y' = 0(15 - 0) - 0 = 0$.
- Substitute into the third equation: $z' = (0)(0) - (8/3)(0) = 0$.
- Since all derivatives are zero, the origin is an equilibrium.

Answer: Yes, $(0, 0, 0)$ is an equilibrium.

2. Find the non-zero equilibrium points for the Lorenz system when $\sigma = 10$, $\beta = 2$, and $\rho = 3$.

- Set $x' = 0 \implies x = y$.
- Set $z' = 0 \implies z = x^2/\beta = x^2/2$.
- Substitute into $y' = 0$: $x(3 - x^2/2) - x = 0$.
- Factor: $x(2 - x^2/2) = 0$.
- Solve $x^2/2 = 2 \implies x^2 = 4 \implies x = \pm 2$.
- For $x = 2$, $y = 2$ and $z = 2^2/2 = 2$. For $x = -2$, $y = -2$ and $z = (-2)^2/2 = 2$.

Answer: $(2, 2, 2)$ and $(-2, -2, 2)$

3. Explain why the Poincare-Bendixson theorem implies that the Lorenz system must have at least three variables to exhibit chaos.

- The Poincare-Bendixson theorem states that in a bounded 2D region, any trajectory that does not approach a fixed point must approach a limit cycle.
- Chaos is characterized by non-periodic, non-convergent trajectories that never intersect.
- In 2D, a trajectory cannot 'dodge' another trajectory or a limit cycle without crossing it, which is forbidden by uniqueness of solutions.
- Therefore, the complex folding and weaving required for a strange attractor requires a third dimension to avoid self-intersection.

Answer: Because 2D bounded trajectories are limited to fixed points or limit cycles; a third dimension is needed to allow non-intersecting, non-periodic paths.