

Bifurcations II — pitchfork and Hopf

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Practice problems

1. For the system $y' = 3y - 2y^3$, find all equilibrium points and determine their stability.
2. In the Hopf normal form with $\mu = 4$, what is the radius of the stable limit cycle, and what is the period of the oscillation?
3. Consider the system $x' = \mu x - y - x(x^2 + y^2)$ and $y' = x + \mu y - y(x^2 + y^2)$. Show that for $\mu = 1$, the trajectory $x = \cos(t), y = \sin(t)$ is a solution.

Solutions

1. For the system $y' = 3y - 2y^3$, find all equilibrium points and determine their stability.

- Set $3y - 2y^3 = 0$, which factors as $y(3 - 2y^2) = 0$.
- The equilibria are $y = 0$ and $y = \pm\sqrt{3/2}$.
- The derivative of the right hand side is $f'(y) = 3 - 6y^2$.
- At $y = 0$, $f'(0) = 3 > 0$, so it is unstable.
- At $y = \pm\sqrt{3/2}$, $f'(\pm\sqrt{3/2}) = 3 - 6(3/2) = 3 - 9 = -6 < 0$, so they are stable.

Answer: $y = 0$ \text{ (unstable)}, $y = \pm\sqrt{1.5}$ \text{ (stable)}

2. In the Hopf normal form with $\mu = 4$, what is the radius of the stable limit cycle, and what is the period of the oscillation?

- The radius of the limit cycle is given by $R = \sqrt{\mu}$.
- Substituting $\mu = 4$ gives $R = \sqrt{4} = 2$.
- The angular velocity is $\theta' = 1$.
- The period T is $2\pi/\theta' = 2\pi/1 = 2\pi$.

Answer: $R = 2$, $T = 2\pi$

3. Consider the system $x' = \mu x - y - x(x^2 + y^2)$ and $y' = x + \mu y - y(x^2 + y^2)$. Show that for $\mu = 1$, the trajectory $x = \cos(t)$, $y = \sin(t)$ is a solution.

- Compute $x' = -\sin(t)$ and $y' = \cos(t)$.
- Substitute into x' equation: $1 \cos(t) - \sin(t) - \cos(t)(\cos^2(t) + \sin^2(t)) = \cos(t) - \sin(t) - \cos(t) = -\sin(t)$.
- Substitute into y' equation: $\cos(t) + 1 \sin(t) - \sin(t)(\cos^2(t) + \sin^2(t)) = \cos(t) + \sin(t) - \sin(t) = \cos(t)$.
- Both equations are satisfied.

Answer: The trajectory satisfies both ODEs exactly.