

Bifurcations I — saddle-node and transcritical

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Practice problems

1. Find the bifurcation point r_c and sketch the bifurcation diagram for the system $y' = r + 2y - y^2$. Identify which branch is stable.
2. Analyze the system $y' = ry - 2y^3$. Does this exhibit a transcritical bifurcation at $r = 0$? Explain why or why not.
3. A population model is given by $y' = ry(1 - y/K) - H$, where H is a constant harvesting rate. For a fixed $K = 100$ and $H = 10$, find the value of r at which a saddle-node bifurcation occurs.

Solutions

1. Find the bifurcation point r_c and sketch the bifurcation diagram for the system $y' = r + 2y - y^2$. Identify which branch is stable.

- Set $r + 2y - y^2 = 0$ and solve for y using the quadratic formula: $y = \frac{-2 \pm \sqrt{4 - 4(-1)(r)}}{-2} = 1 \pm \sqrt{1 + r}$.
- The bifurcation occurs when the discriminant is zero: $1 + r = 0 \implies r_c = -1$.
- For $r > -1$, there are two equilibria: $y = 1 + \sqrt{1 + r}$ and $y = 1 - \sqrt{1 + r}$.
- Check stability using $f'(y) = 2 - 2y$. For the upper branch $y = 1 + \sqrt{1 + r}$, $f'(y) = -2\sqrt{1 + r} < 0$, so it is stable.
- For the lower branch $y = 1 - \sqrt{1 + r}$, $f'(y) = 2\sqrt{1 + r} > 0$, so it is unstable.

Answer: Bifurcation point $r_c = -1$; stable branch $y = 1 + \sqrt{1 + r}$, unstable branch $y = 1 - \sqrt{1 + r}$.

2. Analyze the system $y' = ry - 2y^3$. Does this exhibit a transcritical bifurcation at $r = 0$? Explain why or why not.

- Find equilibria: $y(r - 2y^2) = 0$. This gives $y = 0$ and $y = \pm\sqrt{r/2}$.
- For $r < 0$, only $y = 0$ exists.
- For $r > 0$, three equilibria exist: $y = 0, y = \sqrt{r/2}, y = -\sqrt{r/2}$.
- A transcritical bifurcation requires two equilibria to exist on both sides of the bifurcation point and exchange stability.
- Here, the number of equilibria changes from 1 to 3, which is characteristic of a pitchfork bifurcation, not a transcritical one.

Answer: No, it is a pitchfork bifurcation because the number of equilibria changes from 1 to 3.

3. A population model is given by $y' = ry(1 - y/K) - H$, where H is a constant harvesting rate. For a fixed $K = 100$ and $H = 10$, find the value of r at which a saddle-node bifurcation occurs.

- The equation is $y' = ry - \frac{r}{100}y^2 - 10$.
- Equilibria occur at $-\frac{r}{100}y^2 + ry - 10 = 0$.
- This is a quadratic in y . A saddle-node bifurcation occurs when the discriminant is zero: $D = r^2 - 4(-\frac{r}{100})(-10) = 0$.
- Simplify: $r^2 - \frac{40r}{100} = 0 \implies r^2 - 0.4r = 0$.
- Solving for r (excluding $r = 0$ as it would make the equation $y' = -10$): $r = 0.4$.

Answer: The saddle-node bifurcation occurs at $r = 0.4$.