

The Poincare-Bendixson Theorem

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Practice problems

1. Given a system where $\frac{dr}{dt} = r(1 - r^2)$ and $\frac{d\theta}{dt} = 1$, show that the region $R = \{(r, \theta) : 0.5 \leq r \leq 2\}$ is a trapping region and use the Poincare-Bendixson theorem to conclude a limit cycle exists.
2. Consider a 2D system with a trapping region D . If D contains exactly one fixed point which is an unstable spiral, does the Poincare-Bendixson theorem guarantee a limit cycle?
3. Explain why the Poincare-Bendixson theorem does not apply to the Lorenz system
$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases} .$$

Solutions

1. Given a system where $\frac{dr}{dt} = r(1 - r^2)$ and $\frac{d\theta}{dt} = 1$, show that the region $R = \{(r, \theta) : 0.5 \leq r \leq 2\}$ is a trapping region and use the Poincare-Bendixson theorem to conclude a limit cycle exists.
 - Check the inner boundary $r = 0.5$: $\frac{dr}{dt} = 0.5(1 - 0.25) = 0.375 > 0$, so the flow points outward into R .
 - Check the outer boundary $r = 2$: $\frac{dr}{dt} = 2(1 - 4) = -6 < 0$, so the flow points inward into R .
 - Identify fixed points: $\frac{dr}{dt} = 0$ at $r = 0$ and $r = 1$. $\frac{d\theta}{dt} = 1$ means there are no fixed points since $\frac{d\theta}{dt}$ is never zero.
 - Since R is a bounded trapping region with no fixed points, a limit cycle must exist by Poincare-Bendixson.

Answer: The region is trapping and contains no fixed points, thus a limit cycle exists at $r = 1$.

2. Consider a 2D system with a trapping region D . If D contains exactly one fixed point which is an unstable spiral, does the Poincare-Bendixson theorem guarantee a limit cycle?
 - The theorem requires the region to have no fixed points.
 - However, we can create a smaller trapping region by removing a small disk around the unstable spiral.
 - Since the spiral is unstable, trajectories near it move away from it, meaning the boundary of this small disk acts as an 'outward' boundary.
 - The resulting annular region is now a trapping region with no fixed points.

Answer: Yes, by excluding the unstable fixed point, we create a trapping region without fixed points, guaranteeing a limit cycle.

3. Explain why the Poincare-Bendixson theorem does not apply to the Lorenz system
$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases} .$$
 - The Poincare-Bendixson theorem specifically applies to systems of two first-order ODEs (2D flows).
 - The Lorenz system consists of three first-order ODEs, meaning it is a 3D flow.
 - In 3D, trajectories can avoid crossing without being forced into a point or a loop.

Answer: The theorem only applies to 2D systems; the Lorenz system is 3D.