

Limit cycles; the van der Pol oscillator

Lesson 71 · Module 5 — Nonlinear Dynamics & Chaos · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Consider the system $x' = y$ and $y' = (1 - x^2)y - x$. If the system starts at $(0.01, 0)$, will it converge to the origin or spiral outward? Explain using the concept of damping.
2. For the van der Pol oscillator $y'' - \mu(1 - y^2)y' + y = 0$, what happens to the limit cycle as $\mu \rightarrow 0$? Describe the shape in the phase plane.
3. A system is described by $x' = y$ and $y' = -x + y(x^2 - 1)$. Is the limit cycle in this system stable or unstable? Justify your answer by checking the behavior near the origin.

Solutions

1. Consider the system $x' = y$ and $y' = (1 - x^2)y - x$. If the system starts at $(0.01, 0)$, will it converge to the origin or spiral outward? Explain using the concept of damping.

- Identify the damping term as $(1 - x^2)y$.
- At the starting point $x = 0.01$, the term $(1 - x^2)$ is approximately $1 - 0.0001 = 0.9999$.
- Since the coefficient of y is positive in the y' equation (which corresponds to negative damping in the standard $y'' + cy' + ky = 0$ form), energy is added to the system.
- Therefore, the trajectory will spiral outward away from the origin.

Answer: It will spiral outward because the damping is negative for small x .

2. For the van der Pol oscillator $y'' - \mu(1 - y^2)y' + y = 0$, what happens to the limit cycle as $\mu \rightarrow 0$? Describe the shape in the phase plane.

- As $\mu \rightarrow 0$, the nonlinear damping term $\mu(1 - y^2)y'$ vanishes.
- The equation approaches the linear harmonic oscillator $y'' + y = 0$.
- The trajectories of a linear harmonic oscillator are circles in the phase plane (when scaled).
- Thus, the limit cycle becomes nearly circular and the attraction to it becomes very slow.

Answer: The limit cycle becomes nearly circular, resembling a linear harmonic oscillator.

3. A system is described by $x' = y$ and $y' = -x + y(x^2 - 1)$. Is the limit cycle in this system stable or unstable? Justify your answer by checking the behavior near the origin.

- Look at the damping term $y(x^2 - 1)$.
- Near the origin ($x \approx 0$), the term is approximately $-y$.
- In the equation $y' = -x - y$, the damping is positive, meaning energy is removed.
- Since the origin is stable (trajectories spiral inward), the limit cycle (which exists at $x \approx 1$) must repel trajectories from the inside.
- Therefore, the limit cycle is unstable.

Answer: The limit cycle is unstable because the origin is a stable spiral.