

Energy methods and conserved quantities

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Practice problems

1. Find the conserved quantity $H(x, y)$ for the system $x' = xy$ and $y' = -x^2$.
2. Given the Hamiltonian $H(x, y) = e^x + \frac{1}{2}y^2$, write the corresponding system of first-order ODEs.
3. For the system $x' = y$ and $y' = -x - x^3$, find the conserved quantity $H(x, y)$ and determine if the origin is a center or a saddle.

Solutions

1. Find the conserved quantity $H(x, y)$ for the system $x' = xy$ and $y' = -x^2$.

- Set up the condition for a conserved quantity: $\frac{\partial H}{\partial x}(xy) + \frac{\partial H}{\partial y}(-x^2) = 0$.
- Divide by x (assuming $x \neq 0$): $y \frac{\partial H}{\partial x} - x \frac{\partial H}{\partial y} = 0$.
- This implies the ratio of partials is $\frac{\partial H}{\partial x} / \frac{\partial H}{\partial y} = x/y$.
- Integrating $\frac{\partial H}{\partial x} = x$ gives $H = \frac{1}{2}x^2 + g(y)$.
- Integrating $\frac{\partial H}{\partial y} = y$ gives $H = \frac{1}{2}y^2 + f(x)$.
- Combining these, we find $H(x, y) = \frac{1}{2}(x^2 + y^2)$.

Answer: $H(x, y) = \frac{1}{2}(x^2 + y^2)$

2. Given the Hamiltonian $H(x, y) = e^x + \frac{1}{2}y^2$, write the corresponding system of first-order ODEs.

- Use the Hamiltonian rules: $x' = \frac{\partial H}{\partial y}$ and $y' = -\frac{\partial H}{\partial x}$.
- Compute the partial derivative with respect to y : $\frac{\partial H}{\partial y} = y$.
- Compute the partial derivative with respect to x : $\frac{\partial H}{\partial x} = e^x$.
- Substitute these into the rules to get the system.

Answer: $x' = y, \quad y' = -e^x$

3. For the system $x' = y$ and $y' = -x - x^3$, find the conserved quantity $H(x, y)$ and determine if the origin is a center or a saddle.

- The condition is $\frac{\partial H}{\partial x}(y) + \frac{\partial H}{\partial y}(-x - x^3) = 0$.
- Rearrange to $\frac{\partial H}{\partial x} = x + x^3$ and $\frac{\partial H}{\partial y} = y$.
- Integrate $\frac{\partial H}{\partial x}$ with respect to x : $H = \frac{1}{2}x^2 + \frac{1}{4}x^4 + g(y)$.
- Integrate $\frac{\partial H}{\partial y}$ with respect to y : $H = \frac{1}{2}y^2 + f(x)$.
- The conserved quantity is $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$.
- Since H has a strict local minimum at $(0, 0)$, the level sets are closed loops around the origin, making it a center.

Answer: $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$; the origin is a center.