

Predator-prey (Lotka-Volterra) — full analysis

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Practice problems

1. For the system $\frac{dx}{dt} = 2x - xy$ and $\frac{dy}{dt} = 0.5xy - 3y$, find the non-zero equilibrium point (x^*, y^*) and compute the Jacobian matrix at this point.
2. Using the Jacobian from the previous problem, find the eigenvalues of the coexistence equilibrium and describe the linearised behavior.
3. Show that the function $V(x, y) = 0.5x - 3 \ln x + y - 2 \ln y$ is constant along trajectories of the system $\frac{dx}{dt} = 2x - xy$ and $\frac{dy}{dt} = 0.5xy - 3y$.

Solutions

1. For the system $\frac{dx}{dt} = 2x - xy$ and $\frac{dy}{dt} = 0.5xy - 3y$, find the non-zero equilibrium point (x^*, y^*) and compute the Jacobian matrix at this point.

- Set $2x - xy = 0 \implies x(2 - y) = 0$. Since $x \neq 0$, $y^* = 2$.
- Set $0.5xy - 3y = 0 \implies y(0.5x - 3) = 0$. Since $y \neq 0$, $0.5x = 3 \implies x^* = 6$.
- The Jacobian is $J = \begin{pmatrix} 2 - y & -x \\ 0.5y & 0.5x - 3 \end{pmatrix}$.
- At $(6, 2)$, $J = \begin{pmatrix} 2 - 2 & -6 \\ 0.5(2) & 0.5(6) - 3 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & 0 \end{pmatrix}$.

Answer: $(x^*, y^*) = (6, 2)$, $J = \begin{pmatrix} 0 & -6 \\ 1 & 0 \end{pmatrix}$

2. Using the Jacobian from the previous problem, find the eigenvalues of the coexistence equilibrium and describe the linearised behavior.

- The characteristic equation is $\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & -6 \\ 1 & -\lambda \end{pmatrix} = 0$.
- This gives $\lambda^2 - (-6)(1) = 0 \implies \lambda^2 + 6 = 0$.
- The eigenvalues are $\lambda = \pm i\sqrt{6}$.
- Since the eigenvalues are purely imaginary, the linearised system is a center.

Answer: $\lambda = \pm i\sqrt{6}$, \text{linear centre}

3. Show that the function $V(x, y) = 0.5x - 3 \ln x + y - 2 \ln y$ is constant along trajectories of the system $\frac{dx}{dt} = 2x - xy$ and $\frac{dy}{dt} = 0.5xy - 3y$.

- Compute the total derivative $\frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt}$.
- $\frac{\partial V}{\partial x} = 0.5 - \frac{3}{x}$ and $\frac{\partial V}{\partial y} = 1 - \frac{2}{y}$.
- $\frac{dV}{dt} = (0.5 - \frac{3}{x})(2x - xy) + (1 - \frac{2}{y})(0.5xy - 3y)$.
- $\frac{dV}{dt} = (0.5 - \frac{3}{x})x(2 - y) + (1 - \frac{2}{y})y(0.5x - 3)$.
- $\frac{dV}{dt} = (0.5x - 3)(2 - y) + (y - 2)(0.5x - 3)$.
- $\frac{dV}{dt} = (0.5x - 3)(2 - y) - (2 - y)(0.5x - 3) = 0$.

Answer: $\frac{dV}{dt} = 0$