Linearisation near equilibria; the Jacobian

Practice problems

1. Find the Jacobian matrix for the system $x' = x - x^2 - y$ and $y' = x^2 + y$, and classify the equilibrium point at $(0, 0)$.
2. For the system $x' = y$, $y' = -x - x^3$, find the equilibrium point and determine if the linearisation at that point is conclusive.
3. Consider the system $x' = 2x - y$, $y' = x + 2y - x^2$. Find the equilibrium point in the first quadrant and classify it using the Jacobian.

Solutions

1. Find the Jacobian matrix for the system $x' = x - x^2 - y$ and $y' = x^2 + y$, and classify the equilibrium point at $(0, 0)$.

   - Compute partials: $f_x = 1 - 2x$, $f_y = -1$, $g_x = 2x$, $g_y = 1$.
   - Evaluate at $(0, 0)$: $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.
   - Find eigenvalues: $\det(A - \lambda I) = (1 - \lambda)^2 = 0$, so $\lambda = 1, 1$.
   - Since both eigenvalues are positive, the origin is an unstable node (source).

   Answer: Unstable node (source)

2. For the system $x' = y$, $y' = -x - x^3$, find the equilibrium point and determine if the linearisation at that point is conclusive.

   - Equilibria: $y = 0$ and $-x - x^3 = 0 \implies x(1 + x^2) = 0 \implies x = 0$. Point is $(0, 0)$.
   - Jacobian: $J = \begin{pmatrix} 0 & 1 \\ -1 - 3x^2 & 0 \end{pmatrix}$.
   - Evaluate at $(0, 0)$: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
   - Eigenvalues: $\lambda^2 + 1 = 0 \implies \lambda = \pm i$.
   - Since $\mathrm{Re}(\lambda) = 0$, the linearisation is inconclusive.

   Answer: Inconclusive

3. Consider the system $x' = 2x - y$, $y' = x + 2y - x^2$. Find the equilibrium point in the first quadrant and classify it using the Jacobian.

   - Equilibria: $y = 2x$ and $x + 2(2x) - x^2 = 0 \implies 5x - x^2 = 0 \implies x = 0, 5$. First quadrant point is $(5, 10)$.
   - Jacobian: $J = \begin{pmatrix} 2 & -1 \\ 1 - 2x & 2 \end{pmatrix}$.
   - Evaluate at $(5, 10)$: $A = \begin{pmatrix} 2 & -1 \\ -9 & 2 \end{pmatrix}$.
   - Eigenvalues: $(2 - \lambda)^2 - 9 = 0 \implies 2 - \lambda = \pm 3 \implies \lambda = 5, -1$.
   - Opposite signs indicate a saddle point.

   Answer: Saddle point