

Nonlinear systems: where formulas end and geometry begins

Lesson 65 · Module 5 — Nonlinear Dynamics & Chaos · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Given the nonlinear ODE $y' = y^3$, and two solutions $y_1(t) = (-2t)^{-1/2}$ and $y_2(t) = (-2(t - 1))^{-1/2}$, show that $y_{\text{sum}} = y_1 + y_2$ is not a solution by comparing $(y_{\text{sum}})'$ and $(y_{\text{sum}})^3$.
2. Find all equilibrium points for the nonlinear system
$$\begin{cases} x' = x(1 - x) \\ y' = y(2 - x) \end{cases}.$$
3. Explain why a linear system $\mathbf{x}' = A\mathbf{x}$ cannot have a limit cycle, whereas a nonlinear system can.

Solutions

1. Given the nonlinear ODE $y' = y^3$, and two solutions $y_1(t) = (-2t)^{-1/2}$ and $y_2(t) = (-2(t - 1))^{-1/2}$, show that $y_{\text{sum}} = y_1 + y_2$ is not a solution by comparing $(y_{\text{sum}})'$ and $(y_{\text{sum}})^3$.

- Compute the derivative: $(y_{\text{sum}})' = y_1' + y_2' = y_1^3 + y_2^3$.
- Compute the cube of the sum: $(y_{\text{sum}})^3 = (y_1 + y_2)^3 = y_1^3 + 3y_1^2y_2 + 3y_1y_2^2 + y_2^3$.
- Observe that $(y_{\text{sum}})' - (y_{\text{sum}})^3 = -3y_1^2y_2 - 3y_1y_2^2$, which is not zero.

Answer: The sum is not a solution because the cross-terms in the binomial expansion of $(y_1 + y_2)^3$ prevent the equation from being satisfied.

2. Find all equilibrium points for the nonlinear system $\begin{cases} x' = x(1 - x) \\ y' = y(2 - x) \end{cases}$.

- Set $x' = 0$: $x(1 - x) = 0$ implies $x = 0$ or $x = 1$.
- Case $x = 0$: Substitute into $y' = 0 \implies y(2 - 0) = 0 \implies y = 0$. Point: $(0, 0)$.
- Case $x = 1$: Substitute into $y' = 0 \implies y(2 - 1) = 0 \implies y = 0$. Point: $(1, 0)$.

Answer: The equilibrium points are $(0, 0)$ and $(1, 0)$.

3. Explain why a linear system $\mathbf{x}' = A\mathbf{x}$ cannot have a limit cycle, whereas a nonlinear system can.

- In a linear system, the only possible closed trajectories are centers, where eigenvalues are purely imaginary.
- Centers are not isolated; every nearby trajectory is also a closed loop of a different size.
- A limit cycle must be isolated, meaning nearby trajectories must spiral toward or away from it.
- Linear systems lack the 'saturation' or 'feedback' terms (like x^3 or xy) needed to push trajectories toward a specific closed loop.

Answer: Linear systems only have non-isolated centers; limit cycles require nonlinearity to create isolated, attracting or repelling closed orbits.