

Module 4 Review and Self-Test

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Practice problems

1. Find the general solution and classify the equilibrium point at the origin for the system $\begin{cases} x' = 2x + 3y \\ y' = 4x - y \end{cases}$.
2. Determine the stability and type of the equilibrium for the system $\begin{cases} x' = -x + y \\ y' = -x - y \end{cases}$.
3. Find the steady-state solution (particular solution) for the non-homogeneous system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

Solutions

1. Find the general solution and classify the equilibrium point at the origin for the system $\begin{cases} x' = 2x + 3y \\ y' = 4x - y \end{cases}$.

- Write the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.
- Find the characteristic equation: $\det(A - \lambda I) = (2 - \lambda)(-1 - \lambda) - 12 = \lambda^2 - \lambda - 14 = 0$.
- Solve for λ : $\lambda = \frac{1 \pm \sqrt{1+56}}{2} = \frac{1 \pm \sqrt{57}}{2}$.
- Since the eigenvalues are real and have opposite signs ($\sqrt{57} > 1$), the equilibrium is a saddle point.
- Find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and write $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$.

Answer: Saddle point; $\lambda = \frac{1 \pm \sqrt{57}}{2}$

2. Determine the stability and type of the equilibrium for the system $\begin{cases} x' = -x + y \\ y' = -x - y \end{cases}$.

- Matrix $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.
- Trace $\text{tr}(A) = -2$, Determinant $\det(A) = (-1)(-1) - (1)(-1) = 2$.
- Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0$.
- Eigenvalues: $\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.
- Since the real part is negative and the imaginary part is non-zero, it is a stable spiral.

Answer: Stable spiral

3. Find the steady-state solution (particular solution) for the non-homogeneous system $\begin{pmatrix} x' \\ y' \end{pmatrix} =$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

- Set $\mathbf{x}' = 0$ for the steady state: $0 = A\mathbf{x}_p + \mathbf{b}$.
- Solve the system: $-2x + y = -3$ and $x - 2y = -3$.
- From the first eq, $y = 2x - 3$. Substitute into second: $x - 2(2x - 3) = -3$.
- Simplify: $x - 4x + 6 = -3 \implies -3x = -9 \implies x = 3$.
- Find y : $y = 2(3) - 3 = 3$.

Answer: $\mathbf{x}_p = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$