

Systems clinic — full worked analyses

Lesson 63 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Perform a full analysis of the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Classify the equilibrium, find the general solution, and describe the phase portrait.
2. Given the system $x' = 2x + 3y$ and $y' = 2x + y$, find the eigenvalues and eigenvectors, and determine the type of equilibrium at the origin.
3. A system has a matrix A with $\text{tr}(A) = 0$ and $\det(A) = 9$. Find the eigenvalues and describe the motion of a trajectory starting at $(1, 0)$.

Solutions

1. Perform a full analysis of the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Classify the equilibrium, find the general solution, and describe the phase portrait.
- Compute trace and determinant: $\text{tr}(A) = -4$, $\det(A) = (-1)(-3) - (2)(-1) = 3 + 2 = 5$.
 - Solve characteristic equation: $\lambda^2 + 4\lambda + 5 = 0$. Using quadratic formula, $\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$.
 - Since the real part is -2 and the imaginary part is 1 , the equilibrium is a stable spiral.
 - The general solution takes the form $\mathbf{x}(t) = e^{-2t}(c_1 \mathbf{u}(t) + c_2 \mathbf{w}(t))$ where $\mathbf{u}$ and $\mathbf{w}$ are trigonometric vectors.

Answer: Stable spiral with eigenvalues $\lambda = -2 \pm i$.

2. Given the system $x' = 2x + 3y$ and $y' = 2x + y$, find the eigenvalues and eigenvectors, and determine the type of equilibrium at the origin.
- Matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.
 - Characteristic equation: $\det \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$.
 - Factoring gives $(\lambda - 4)(\lambda + 1) = 0$, so $\lambda_1 = 4$, $\lambda_2 = -1$.
 - For $\lambda = 4$: $-2x + 3y = 0 \implies \mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
 - For $\lambda = -1$: $3x + 3y = 0 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
 - Since eigenvalues are real and opposite, the origin is a saddle point.

Answer: Saddle point with $\lambda = 4, -1$ and eigenvectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. A system has a matrix A with $\text{tr}(A) = 0$ and $\det(A) = 9$. Find the eigenvalues and describe the motion of a trajectory starting at $(1, 0)$.
- Characteristic equation: $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0 \implies \lambda^2 + 9 = 0$.
 - Eigenvalues are $\lambda = \pm 3i$.
 - This is a center. The trajectories are ellipses centered at the origin.
 - The motion is a periodic orbit with frequency $\omega = 3$ and period $T = 2\pi/3$.

Answer: Center with eigenvalues $\lambda = \pm 3i$; the trajectory is a periodic elliptical orbit.