

# Compartment models: drug dosing

Lesson 62 · Module 4 — Systems of ODEs · Differential Equations · [dailymaths.uk/diffeq](http://dailymaths.uk/diffeq)

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## Practice problems

1. A drug is modeled by the system  $x' = -4x + 2y$  and  $y' = 3x - 3y$ . Find the eigenvalues of the system and determine the half-life of the slow mode.
2. For the system  $x' = -3x + y$ ,  $y' = 2x - 2y$  with initial conditions  $x(0) = 1$ ,  $y(0) = 0$ , find the amount of drug in the tissue  $y(t)$  at time  $t$ .
3. A patient requires a drug to be maintained at a level where the slow-mode component does not drop below 50% of its initial value. If the system eigenvalues are  $-0.2$  and  $-1.5$ , what is the maximum allowable time between doses?

## Solutions

1. A drug is modeled by the system  $x' = -4x + 2y$  and  $y' = 3x - 3y$ . Find the eigenvalues of the system and determine the half-life of the slow mode.

- The matrix is  $A = \begin{pmatrix} -4 & 2 \\ 3 & -3 \end{pmatrix}$ .
- The characteristic equation is  $(-4 - \lambda)(-3 - \lambda) - 6 = 0$ , which is  $\lambda^2 + 7\lambda + 12 - 6 = \lambda^2 + 7\lambda + 6 = 0$ .
- Factoring gives  $(\lambda + 1)(\lambda + 6) = 0$ , so  $\lambda_1 = -1$  and  $\lambda_2 = -6$ .
- The slow eigenvalue is  $\lambda_{\text{slow}} = -1$ .
- The half-life is  $t_{1/2} = \ln 2 / |-1| \approx 0.693$ .

**Answer:**  $t_{1/2} \approx 0.693$

2. For the system  $x' = -3x + y$ ,  $y' = 2x - 2y$  with initial conditions  $x(0) = 1$ ,  $y(0) = 0$ , find the amount of drug in the tissue  $y(t)$  at time  $t$ .

- Eigenvalues are  $\lambda_1 = -1$ ,  $\lambda_2 = -4$  with eigenvectors  $\mathbf{v}_1 = (1, 2)^T$ ,  $\mathbf{v}_2 = (1, -1)^T$ .
- General solution:  $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .
- Initial conditions:  $c_1 + c_2 = 1$  and  $2c_1 - c_2 = 0$ .
- Adding equations:  $3c_1 = 1 \implies c_1 = 1/3$ . Then  $c_2 = 2/3$ .
- The tissue amount is  $y(t) = 2(1/3)e^{-t} - (2/3)e^{-4t} = \frac{2}{3}(e^{-t} - e^{-4t})$ .

**Answer:**  $y(t) = \frac{2}{3}(e^{-t} - e^{-4t})$

3. A patient requires a drug to be maintained at a level where the slow-mode component does not drop below 50% of its initial value. If the system eigenvalues are  $-0.2$  and  $-1.5$ , what is the maximum allowable time between doses?

- The slow eigenvalue is  $\lambda_{\text{slow}} = -0.2$ .
- The slow-mode component decays as  $e^{-0.2t}$ .
- We want  $e^{-0.2t} = 0.5$ .
- Taking the natural log:  $-0.2t = \ln(0.5) = -\ln 2$ .
- Solving for  $t$ :  $t = \ln 2 / 0.2 = 0.693 / 0.2 = 3.465$ .

**Answer:**  $t \approx 3.465$