

Coupled Oscillators and Normal Modes

Lesson 61 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Two masses of 1 kg are connected by springs such that the stiffness matrix is $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$. Find the natural frequencies ω_1, ω_2 and the corresponding normal mode eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.
2. For a system with $M = I$ and $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, find the general solution $\mathbf{x}(t)$ given the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
3. Consider three identical masses m connected by four identical springs k (fixed at both ends). The stiffness matrix is $K = \frac{k}{m} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find the three natural frequencies of this system.

Solutions

1. Two masses of 1 kg are connected by springs such that the stiffness matrix is $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$. Find the natural frequencies ω_1, ω_2 and the corresponding normal mode eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

- Solve $\det(K - \lambda I) = (3 - \lambda)(2 - \lambda) - 1 = \lambda^2 - 5\lambda + 5 = 0$.
- Use the quadratic formula: $\lambda = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}$.
- The frequencies are $\omega_1 = \sqrt{\frac{5 - \sqrt{5}}{2}}$ and $\omega_2 = \sqrt{\frac{5 + \sqrt{5}}{2}}$.
- Find eigenvectors by solving $(K - \lambda I)\mathbf{v} = 0$ for each λ .

Answer: $\omega_1 = \sqrt{1.38}$, $\omega_2 = \sqrt{3.62}$, $\mathbf{v}_1 \approx \begin{pmatrix} 1 \\ 1.62 \end{pmatrix}$, $\mathbf{v}_2 \approx \begin{pmatrix} 1 \\ -0.62 \end{pmatrix}$

2. For a system with $M = I$ and $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, find the general solution $\mathbf{x}(t)$ given the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- From the lesson, eigenvalues are $\lambda_1 = 1, \lambda_2 = 3$ with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Set $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Solving the system: $c_1 + c_2 = 0$ and $c_1 - c_2 = 1$ gives $c_1 = 1/2, c_2 = -1/2$.
- The general solution is $\mathbf{x}(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(t) - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}t)$.

Answer: $\mathbf{x}(t) = \frac{1}{2} \begin{pmatrix} \cos(t) - \cos(\sqrt{3}t) \\ \cos(t) + \cos(\sqrt{3}t) \end{pmatrix}$

3. Consider three identical masses m connected by four identical springs k (fixed at both ends). The stiffness matrix is $K = \frac{k}{m} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find the three natural frequencies of this system.

- Solve $\det(K - \lambda I) = 0$ where K is the matrix above (setting $k/m = 1$ for simplicity).
- The determinant is $(2 - \lambda)((2 - \lambda)^2 - 1) - (-1)(-(2 - \lambda)) = (2 - \lambda)((2 - \lambda)^2 - 2) = 0$.
- The roots are $\lambda_1 = 2, \lambda_2 = 2 - \sqrt{2},$ and $\lambda_3 = 2 + \sqrt{2}$.
- The frequencies are $\omega_1 = \sqrt{2k/m}, \omega_2 = \sqrt{(2 - \sqrt{2})k/m},$ and $\omega_3 = \sqrt{(2 + \sqrt{2})k/m}.$

Answer: $\omega = \sqrt{\frac{k}{m}(2 - \sqrt{2})}, \sqrt{\frac{2k}{m}}, \sqrt{\frac{k}{m}(2 + \sqrt{2})}$