

Nonhomogeneous Systems

Lesson 60 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Find the particular solution $\mathbf{x}_p$ for the system $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.
2. Given the system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \cos(2t) \end{pmatrix}$, find a particular solution using the form $\mathbf{x}_p = \begin{pmatrix} a \sin(2t) + b \cos(2t) \\ c \sin(2t) + d \cos(2t) \end{pmatrix}$.
3. Solve the IVP $\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solutions

1. Find the particular solution $\mathbf{x}_p$ for the system $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

- Set $\mathbf{x}' = \mathbf{0}$ for the particular solution: $\begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$.
- From the second row: $-3y = -2 \implies y = 2/3$.
- From the first row: $-x + 2(2/3) = -5 \implies -x + 4/3 = -15/3 \implies -x = -19/3 \implies x = 19/3$.

Answer: $\mathbf{x}_p = \begin{pmatrix} 19/3 \\ 2/3 \end{pmatrix}$

2. Given the system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \cos(2t) \end{pmatrix}$, find a particular solution using the form $\mathbf{x}_p = \begin{pmatrix} a \sin(2t) + b \cos(2t) \\ c \sin(2t) + d \cos(2t) \end{pmatrix}$.

- Substitute $x = a \sin(2t) + b \cos(2t)$ and $y = c \sin(2t) + d \cos(2t)$ into the system.
- From $x' = y$, we have $2a \cos(2t) - 2b \sin(2t) = c \sin(2t) + d \cos(2t)$, so $d = 2a$ and $c = -2b$.
- From $y' = -x + 2 \cos(2t)$, we have $2c \cos(2t) - 2d \sin(2t) = -a \sin(2t) - b \cos(2t) + 2 \cos(2t)$.
- Substitute $c = -2b$ and $d = 2a$: $-4b \cos(2t) - 4a \sin(2t) = -a \sin(2t) - b \cos(2t) + 2 \cos(2t)$.
- Matching $\sin(2t)$: $-4a = -a \implies a = 0$.
- Matching $\cos(2t)$: $-4b = -b + 2 \implies -3b = 2 \implies b = -2/3$.
- Then $d = 2(0) = 0$ and $c = -2(-2/3) = 4/3$.

Answer: $\mathbf{x}_p(t) = \begin{pmatrix} -\frac{2}{3} \cos(2t) \\ \frac{4}{3} \sin(2t) \end{pmatrix}$

3. Solve the IVP $\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- Find $\mathbf{x}_p$: $-2x + 1 = 0 \implies x = 1/2$ and $-y + 1 = 0 \implies y = 1$. So $\mathbf{x}_p = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.
- Homogeneous solution: $\mathbf{x}_h = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- General solution: $\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.
- Apply ICs: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 1/2 \\ c_2 + 1 \end{pmatrix} \implies c_1 = -1/2, c_2 = -1$.

Answer: $\mathbf{x}(t) = \begin{pmatrix} \frac{1}{2}(1 - e^{-2t}) \\ 1 - e^{-t} \end{pmatrix}$