

The matrix exponential e^{At}

Lesson 59 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Compute the matrix exponential e^{At} for the diagonal matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.
2. Given $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find e^{At} using the diagonalisation method.
3. Solve the system $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and initial conditions $x(0) = 3, y(0) = 5$.

Solutions

1. Compute the matrix exponential e^{At} for the diagonal matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

- Since A is diagonal, A^k is simply the matrix with entries 2^k and $(-3)^k$ on the diagonal.
- The series definition $e^{At} = \sum \frac{(At)^k}{k!}$ applies to each diagonal entry independently.
- The top-left entry is $\sum \frac{(2t)^k}{k!} = e^{2t}$.
- The bottom-right entry is $\sum \frac{(-3t)^k}{k!} = e^{-3t}$.

Answer: $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$

2. Given $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find e^{At} using the diagonalisation method.

- Find eigenvalues: $\det(A - \lambda I) = \lambda^2 - 1 = 0$, so $\lambda = 1, -1$.
- Find eigenvectors: for $\lambda = 1$, $v_1 = [1, 1]^T$; for $\lambda = -1$, $v_2 = [1, -1]^T$.
- Construct $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
- Compute $e^{At} = P \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} P^{-1}$.
- Multiply: $e^{At} = \frac{1}{2} \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$.

Answer: $e^{At} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$

3. Solve the system $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and initial conditions $x(0) = 3, y(0) = 5$.

- The matrix A is diagonal, so $e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$.
- The solution is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.
- Substitute the values: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.
- Perform the multiplication: $x(t) = 3e^t$ and $y(t) = 5e^{2t}$.

Answer: $\mathbf{x}(t) = \begin{bmatrix} 3e^t \\ 5e^{2t} \end{bmatrix}$