

The trace-determinant plane — one diagram to rule them all

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Practice problems

1. Classify the equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$ using the trace-determinant plane.
2. Determine the type and stability of the equilibrium for $A = \begin{pmatrix} -3 & 1 \\ 2 & -3 \end{pmatrix}$.
3. Find the range of values for k such that the system with $A = \begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$ is a stable spiral.

Solutions

1. Classify the equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$ using the trace-determinant plane.

- Compute the trace: $\text{tr}(A) = 2 + (-4) = -2$.
- Compute the determinant: $\det(A) = (2)(-4) - (-1)(3) = -8 + 3 = -5$.
- Since $\det(A) < 0$, the equilibrium is a saddle point.

Answer: Saddle point

2. Determine the type and stability of the equilibrium for $A = \begin{pmatrix} -3 & 1 \\ 2 & -3 \end{pmatrix}$.

- Compute the trace: $\text{tr}(A) = -3 + (-3) = -6$.
- Compute the determinant: $\det(A) = (-3)(-3) - (1)(2) = 9 - 2 = 7$.
- Since $\det(A) > 0$ and $\text{tr}(A) < 0$, the system is stable.
- Compute the discriminant: $\Delta = (-6)^2 - 4(7) = 36 - 28 = 8$.
- Since $\Delta > 0$, the equilibrium is a node.

Answer: Stable node

3. Find the range of values for k such that the system with $A = \begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$ is a stable spiral.

- Compute the trace: $\text{tr}(A) = k + k = 2k$.
- Compute the determinant: $\det(A) = (k)(k) - (1)(-1) = k^2 + 1$.
- For a spiral, we need $\Delta < 0$: $(2k)^2 - 4(k^2 + 1) = 4k^2 - 4k^2 - 4 = -4$. This is always negative, so it is always a spiral.
- For stability, we need $\text{tr}(A) < 0$: $2k < 0$, which means $k < 0$.
- Also need $\det(A) > 0$: $k^2 + 1 > 0$, which is always true.

Answer: $k < 0$