

Repeated Eigenvalues and Defective Matrices

Lesson 56 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.
2. Given a system with a repeated eigenvalue $\lambda = -5$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, find the general solution if the generalised eigenvector is $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
3. Determine if the system $\mathbf{x}' = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x}$ is defective and describe the resulting phase portrait.

Solutions

- Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.
 - Find eigenvalues: $\det(A - \lambda I) = (2 - \lambda)^2 = 0$, so $\lambda = 2$ repeated.
 - Find eigenvector $\mathbf{v}$: $(A - 2I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which implies $v_2 = 0$. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
 - Find generalised eigenvector $\mathbf{w}$: $(A - 2I)\mathbf{w} = \mathbf{v} \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which implies $w_2 = 1$. Let $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
 - Construct general solution: $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} (t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix})$.

Answer: $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

- Given a system with a repeated eigenvalue $\lambda = -5$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, find the general solution if the generalised eigenvector is $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
 - The first solution is $\mathbf{x}_1(t) = e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
 - The second solution is $\mathbf{x}_2(t) = t e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{-5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
 - The general solution is the linear combination of these two.

Answer: $\mathbf{x}(t) = c_1 e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} t+1 \\ -2t \end{pmatrix}$

- Determine if the system $\mathbf{x}' = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x}$ is defective and describe the resulting phase portrait.
 - Find eigenvalues: $\det(A - \lambda I) = (4 - \lambda)^2 = 0$, so $\lambda = 4$ repeated.
 - Find eigenvectors: $(A - 4I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Any vector in $\mathbb{R}^2$ is an eigenvector.
 - Since we have two linearly independent eigenvectors (e.g., $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$), the matrix is NOT defective.
 - Because $\lambda = 4 > 0$ and it is a non-defective repeated eigenvalue, the origin is an unstable star node.

Answer: Not defective; unstable star node.