Solving by eigenvalues II — saddles and nodes

Practice problems

1. Classify the equilibrium point at the origin and sketch the phase portrait for the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
2. Determine the type of equilibrium point for the system $x' = 2x + 3y, y' = 2x + y$.
3. A system has eigenvalues $\lambda_1 = 0.1$ and $\lambda_2 = 10$. Describe the behavior of trajectories far from the origin compared to trajectories very close to the origin.

Solutions

1. Classify the equilibrium point at the origin and sketch the phase portrait for the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

   - Find eigenvalues: $\det(A - \lambda I) = (-3 - \lambda)^2 - 4 = \lambda^2 + 6\lambda + 5 = 0$.
   - Solve for λ: $(\lambda + 1)(\lambda + 5) = 0$, so $\lambda_1 = -1, \lambda_2 = -5$.
   - Since both are negative, the origin is a stable node.
   - Find eigenvectors: For $\lambda = -1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = -5$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   - Trajectories enter the origin tangent to $\mathbf{v}_1$ because -1 is closer to zero than -5.

   Answer: Stable Node; trajectories approach along the line $y = x$.

2. Determine the type of equilibrium point for the system $x' = 2x + 3y, y' = 2x + y$.

   - Matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.
   - Characteristic equation: $(2 - \lambda)(1 - \lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$.
   - Solve for λ: $(\lambda - 4)(\lambda + 1) = 0$, so $\lambda_1 = 4, \lambda_2 = -1$.
   - Since the eigenvalues have opposite signs, the origin is a saddle point.

   Answer: Saddle Point

3. A system has eigenvalues $\lambda_1 = 0.1$ and $\lambda_2 = 10$. Describe the behavior of trajectories far from the origin compared to trajectories very close to the origin.

   - Both eigenvalues are positive, so the origin is an unstable node (source).
   - The general solution is $\mathbf{x}(t) = c_1 e^{0.1t}\mathbf{v}_1 + c_2 e^{10t}\mathbf{v}_2$.
   - Close to the origin (t is very negative), the e^{10t} term is negligible, so trajectories emerge tangent to $\mathbf{v}_1$.
   - Far from the origin (large t), the e^{10t} term completely dominates the $e^{0.1t}$ term, so all trajectories eventually become parallel to the $\mathbf{v}_2$ axis.

   Answer: Unstable node; trajectories emerge tangent to $\mathbf{v}_1$ and eventually align with $\mathbf{v}_2$.