Solving by eigenvalues I — real distinct eigenvalues

Practice problems

1. Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

2. Solve the IVP $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

3. A system has eigenvalues $\lambda_1 = 5, \lambda_2 = -3$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the solution that satisfies $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solutions

1. Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

   - Find eigenvalues: $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = 0$, so $\lambda_1 = 1, \lambda_2 = 2$.
   - Find eigenvector for $\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\mathbf{v} = 0 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   - Find eigenvector for $\lambda = 2$: $\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{v} = 0 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
   - Assemble general solution: $\mathbf{x}(t) = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   Answer: \mathbfx(t) = c_1 e^t \beginpmatrix 1 \\ 1 \endpmatrix + c_2 e^2t \beginpmatrix 1 \\ 2 \endpmatrix

2. Solve the IVP $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

   - Eigenvalues: $(1 - \lambda)^2 - 4 = 0 \implies \lambda = 3, -1$.
   - Eigenvectors: For $\lambda = 3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = -1, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   - General solution: $\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   - Initial conditions: $c_1 + c_2 = 3$ and $c_1 - c_2 = 1$. Solving gives $c_1 = 2, c_2 = 1$.

   Answer: \mathbfx(t) = 2 e^3t \beginpmatrix 1 \\ 1 \endpmatrix + e^-t \beginpmatrix 1 \\ -1 \endpmatrix

3. A system has eigenvalues $\lambda_1 = 5, \lambda_2 = -3$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the solution that satisfies $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

   - General solution: $\mathbf{x}(t) = c_1 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
   - At $t = 0$: $2c_1 + 0c_2 = 4 \implies c_1 = 2$.
   - At $t = 0$: $1c_1 + 1c_2 = 5 \implies 2 + c_2 = 5 \implies c_2 = 3$.
   - Substitute constants back into the general solution.

   Answer: \mathbfx(t) = 2 e^5t \beginpmatrix 2 \\ 1 \endpmatrix + 3 e^-3t \beginpmatrix 0 \\ 1 \endpmatrix

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