

Linear systems in matrix form

Lesson 52 · Module 4 — Systems of ODEs · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Find the general solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
2. For the system $x' = 2x + y, y' = x + 2y$, find the eigenvalues and eigenvectors, and describe the stability of the origin.
3. Solve the IVP $x' = x + 3y, y' = 3x + y$ with $x(0) = 1, y(0) = 1$.

Solutions

1. Find the general solution to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

- Find eigenvalues: $\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$.
- Solve for λ : $(\lambda - 4)(\lambda + 1) = 0$, so $\lambda_1 = 4, \lambda_2 = -1$.
- For $\lambda = 4$: $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \mathbf{v} = 0 \implies 3v_1 = 2v_2$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
- For $\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \mathbf{v} = 0 \implies v_1 = -v_2$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Answer: $\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. For the system $x' = 2x + y, y' = x + 2y$, find the eigenvalues and eigenvectors, and describe the stability of the origin.

- Characteristic equation: $(2 - \lambda)^2 - 1 = 0$, so $\lambda^2 - 4\lambda + 3 = 0$.
- Eigenvalues: $(\lambda - 3)(\lambda - 1) = 0$, so $\lambda_1 = 3, \lambda_2 = 1$.
- For $\lambda = 3$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{v} = 0 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda = 1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{v} = 0 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Since both eigenvalues are positive, the origin is an unstable source.

Answer: $\lambda_1 = 3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \lambda_2 = 1, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \text{Unstable Source}$

3. Solve the IVP $x' = x + 3y, y' = 3x + y$ with $x(0) = 1, y(0) = 1$.

- From the lesson, eigenvalues are 4 and -2 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- General solution: $\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Initial conditions: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- This gives $c_1 + c_2 = 1$ and $c_1 - c_2 = 1$. Adding them gives $2c_1 = 2 \implies c_1 = 1$, so $c_2 = 0$.

Answer: $x(t) = e^{4t}, y(t) = e^{4t}$