

Transfer functions — a systems-theory doorway

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Practice problems

1. Find the transfer function $H(s)$ for the system $y'' + 4y' + 13y = x(t)$ with zero initial conditions. Identify the poles and state whether the system is stable, unstable, or purely oscillatory.
2. A system has a transfer function $H(s) = \frac{s+2}{(s-1)(s+3)}$. If the input is a unit step $x(t) = 1$, find the output $y(t)$.
3. Explain why a system with poles at $s = 0$ and $s = -5$ is considered 'marginally stable' compared to a system with poles at $s = -1$ and $s = -5$.

Solutions

1. Find the transfer function $H(s)$ for the system $y'' + 4y' + 13y = x(t)$ with zero initial conditions. Identify the poles and state whether the system is stable, unstable, or purely oscillatory.

- Take the Laplace transform: $s^2Y(s) + 4sY(s) + 13Y(s) = X(s)$.
- Factor out $Y(s)$: $(s^2 + 4s + 13)Y(s) = X(s)$.
- Form the ratio: $H(s) = \frac{1}{s^2 + 4s + 13}$.
- Find poles using the quadratic formula: $s = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$.
- Since the real part is -2 (negative), the system is stable and exhibits decaying oscillations.

Answer: $H(s) = \frac{1}{s^2 + 4s + 13}$, \text{poles at } s = -2 \pm 3i, \text{stable decaying oscillation}.

2. A system has a transfer function $H(s) = \frac{s+2}{(s-1)(s+3)}$. If the input is a unit step $x(t) = 1$, find the output $y(t)$.

- Input transform is $X(s) = \frac{1}{s}$.
- Output transform is $Y(s) = \frac{s+2}{s(s-1)(s+3)}$.
- Partial fraction decomposition: $\frac{s+2}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$.
- Solving for constants: $A = \frac{2}{(-1)(3)} = -\frac{2}{3}$, $B = \frac{3}{(1)(4)} = \frac{3}{4}$, $C = \frac{-1}{(-3)(-4)} = -\frac{1}{12}$.
- Inverse transform: $y(t) = -\frac{2}{3} + \frac{3}{4}e^t - \frac{1}{12}e^{-3t}$.

Answer: $y(t) = -\frac{2}{3} + \frac{3}{4}e^t - \frac{1}{12}e^{-3t}$

3. Explain why a system with poles at $s = 0$ and $s = -5$ is considered 'marginally stable' compared to a system with poles at $s = -1$ and $s = -5$.

- A pole at $s = -5$ gives a term e^{-5t} , which decays to zero.
- A pole at $s = -1$ gives a term e^{-t} , which also decays to zero.
- A pole at $s = 0$ gives a term $e^{0t} = 1$, which is a constant.
- In the first system, the output does not return to zero but stays at a constant value (the steady-state response).
- In the second system, the output returns completely to zero. Thus, the first is marginally stable.

Answer: The pole at $s = 0$ creates a constant non-decaying term, preventing the system from returning to the zero equilibrium.