

Convolution II — the convolution theorem

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Practice problems

1. Use the convolution theorem to find the inverse Laplace transform of $Y(s) = \frac{1}{(s-2)(s-3)}$.
2. Find the inverse Laplace transform of $Y(s) = \frac{1}{s^2(s-1)}$ using the convolution theorem.
3. A system has a transfer function $H(s) = \frac{1}{s^2+4}$ and is given an input $f(t) = \sin(2t)$. Find the output $y(t)$ using the convolution theorem.

Solutions

1. Use the convolution theorem to find the inverse Laplace transform of $Y(s) = \frac{1}{(s-2)(s-3)}$.

- Identify $F(s) = \frac{1}{s-2}$ and $G(s) = \frac{1}{s-3}$, so $f(t) = e^{2t}$ and $g(t) = e^{3t}$.
- Set up the convolution integral: $y(t) = \int_0^t e^{2\tau} e^{3(t-\tau)} d\tau$.
- Simplify the integrand: $y(t) = e^{3t} \int_0^t e^{2\tau} e^{-3\tau} d\tau = e^{3t} \int_0^t e^{-\tau} d\tau$.
- Integrate: $y(t) = e^{3t} [-e^{-\tau}]_0^t = e^{3t} (-e^{-t} + 1)$.
- Distribute: $y(t) = -e^{2t} + e^{3t}$.

Answer: $y(t) = e^{3t} - e^{2t}$

2. Find the inverse Laplace transform of $Y(s) = \frac{1}{s^2(s-1)}$ using the convolution theorem.

- Identify $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{1}{s-1}$, so $f(t) = t$ and $g(t) = e^t$.
- Set up the integral: $y(t) = \int_0^t \tau e^{t-\tau} d\tau$.
- Pull out e^t : $y(t) = e^t \int_0^t \tau e^{-\tau} d\tau$.
- Use integration by parts on $\int \tau e^{-\tau} d\tau$: $u = \tau, dv = e^{-\tau} d\tau \implies du = d\tau, v = -e^{-\tau}$.
- The integral is $[-\tau e^{-\tau}]_0^t + \int_0^t e^{-\tau} d\tau = -te^{-t} + [-e^{-\tau}]_0^t = -te^{-t} - e^{-t} + 1$.
- Multiply by e^t : $y(t) = e^t(1 - e^{-t} - te^{-t}) = e^t - 1 - t$.

Answer: $y(t) = e^t - t - 1$

3. A system has a transfer function $H(s) = \frac{1}{s^2+4}$ and is given an input $f(t) = \sin(2t)$. Find the output $y(t)$ using the convolution theorem.

- The output in s-domain is $Y(s) = H(s)F(s) = \frac{1}{s^2+4} \cdot \frac{2}{s^2+4} = \frac{2}{(s^2+4)^2}$.
- Identify $h(t) = \frac{1}{2} \sin(2t)$ and $f(t) = \sin(2t)$.
- Set up convolution: $y(t) = \int_0^t \frac{1}{2} \sin(2\tau) \sin(2(t-\tau)) d\tau$.
- Use the product-to-sum identity $\sin(A) \sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$.
- Here $A = 2\tau$ and $B = 2t - 2\tau$, so $A - B = 4\tau - 2t$ and $A + B = 2t$.
- The integral becomes $\frac{1}{4} \int_0^t (\cos(4\tau - 2t) - \cos(2t)) d\tau$.
- Integrate: $\frac{1}{4} [\frac{1}{4} \sin(4\tau - 2t) - \tau \cos(2t)]_0^t$.
- Evaluate: $\frac{1}{4} (\frac{1}{4} \sin(2t) - t \cos(2t) - (\frac{1}{4} \sin(-2t) - 0))$.
- Since $\sin(-2t) = -\sin(2t)$, we get $\frac{1}{4} (\frac{1}{2} \sin(2t) - t \cos(2t))$.

Answer: $y(t) = \frac{1}{8} \sin(2t) - \frac{1}{4} t \cos(2t)$