

Convolution I — the meaning

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Practice problems

1. Compute the convolution $y(t) = (f * h)(t)$ where $f(t) = e^{-t}$ and $h(t) = e^{-3t}$ for $t \geq 0$.
2. A system has an impulse response $h(t) = t$. Find the output $y(t)$ when the input is a constant $f(t) = 5$.
3. Find the convolution of $f(t) = \sin(t)$ and $h(t) = 1$. Interpret the result in terms of the area under the sine curve.

Solutions

1. Compute the convolution $y(t) = (f * h)(t)$ where $f(t) = e^{-t}$ and $h(t) = e^{-3t}$ for $t \geq 0$.

- Set up the integral: $y(t) = \int_0^t e^{-\tau} e^{-3(t-\tau)} d\tau$.
- Simplify the exponent: $y(t) = \int_0^t e^{-\tau} e^{-3t} e^{3\tau} d\tau = e^{-3t} \int_0^t e^{2\tau} d\tau$.
- Integrate: $y(t) = e^{-3t} [\frac{1}{2} e^{2\tau}]_0^t$.
- Evaluate limits: $y(t) = e^{-3t} (\frac{1}{2} e^{2t} - \frac{1}{2})$.
- Distribute: $y(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}$.

Answer: $y(t) = \frac{1}{2}(e^{-t} - e^{-3t})$

2. A system has an impulse response $h(t) = t$. Find the output $y(t)$ when the input is a constant $f(t) = 5$.

- Set up the integral: $y(t) = \int_0^t 5 \cdot (t - \tau) d\tau$.
- Split the integral: $y(t) = 5 \int_0^t t d\tau - 5 \int_0^t \tau d\tau$.
- Integrate with respect to τ : $y(t) = 5[t\tau]_0^t - 5[\frac{1}{2}\tau^2]_0^t$.
- Evaluate: $y(t) = 5t^2 - \frac{5}{2}t^2$.
- Simplify: $y(t) = \frac{5}{2}t^2$.

Answer: $y(t) = \frac{5}{2}t^2$

3. Find the convolution of $f(t) = \sin(t)$ and $h(t) = 1$. Interpret the result in terms of the area under the sine curve.

- Set up the integral: $y(t) = \int_0^t \sin(\tau) \cdot 1 d\tau$.
- Integrate: $y(t) = [-\cos(\tau)]_0^t$.
- Evaluate: $y(t) = -\cos(t) - (-\cos(0))$.
- Simplify: $y(t) = 1 - \cos(t)$.
- Interpretation: The output is the running integral (accumulated area) of the input signal.

Answer: $y(t) = 1 - \cos(t)$