

The Dirac delta — hitting a system with a hammer

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Practice problems

1. Solve the IVP $y'' + 16y = \delta(t - 3)$ with $y(0) = 0$ and $y'(0) = 0$.
2. Find the solution to $y'' + 2y' + 2y = \delta(t - 2)$ with $y(0) = 0$ and $y'(0) = 0$.
3. A system is described by $y'' + 25y = 2\delta(t - 1)$ with $y(0) = 0, y'(0) = 0$. Find $y(t)$.

Solutions

1. Solve the IVP $y'' + 16y = \delta(t - 3)$ with $y(0) = 0$ and $y'(0) = 0$.

- Transform the equation: $(s^2 + 16)Y(s) = e^{-3s}$.
- Solve for $Y(s)$: $Y(s) = e^{-3s} \frac{1}{s^2 + 16}$.
- Identify the inverse transform of $\frac{1}{s^2 + 16}$ as $\frac{1}{4} \sin(4t)$.
- Apply the second shifting theorem to account for e^{-3s} .

Answer: $y(t) = \frac{1}{4} \sin(4(t-3))u(t-3)$

2. Find the solution to $y'' + 2y' + 2y = \delta(t - 2)$ with $y(0) = 0$ and $y'(0) = 0$.

- Transform: $(s^2 + 2s + 2)Y(s) = e^{-2s}$.
- Complete the square: $Y(s) = \frac{e^{-2s}}{(s+1)^2 + 1}$.
- The inverse transform of $\frac{1}{(s+1)^2 + 1}$ is $e^{-t} \sin(t)$.
- Apply the shift $t \rightarrow t - 2$ and multiply by $u(t - 2)$.

Answer: $y(t) = e^{-(t-2)} \sin(t-2)u(t-2)$

3. A system is described by $y'' + 25y = 2\delta(t - 1)$ with $y(0) = 0, y'(0) = 0$. Find $y(t)$.

- Transform: $(s^2 + 25)Y(s) = 2e^{-s}$.
- Solve for $Y(s)$: $Y(s) = 2e^{-s} \frac{1}{s^2 + 25}$.
- The inverse transform of $\frac{1}{s^2 + 25}$ is $\frac{1}{5} \sin(5t)$.
- Multiply by the constant 2 and apply the shift $t \rightarrow t - 1$.

Answer: $y(t) = \frac{2}{5} \sin(5(t-1))u(t-1)$