Discontinuous forcing: step functions

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Practice problems

1. Find the Laplace transform of the function $f(t) = 0$ for $t < 3$ and $f(t) = e^{2t}$ for $t \geq 3$.
2. Solve the IVP $y' + 2y = u(t-1)$ with $y(0) = 0$.
3. Find the inverse Laplace transform of $H(s) = \frac{e^{-2s}}{s^2+9}$.

Solutions

1. Find the Laplace transform of the function $f(t) = 0$ for $t < 3$ and $f(t) = e^{2t}$ for $t \geq 3$.
   - Express the function using the Heaviside step function: $f(t) = u(t-3)e^{2t}$.
   - To use the second shifting theorem, we need the form $u(t-a)g(t-a)$. We rewrite e^{2t} as $e^{2(t-3+3)} = e^{2(t-3)}e^{6}$.
   - Now we have $f(t) = e^{6}u(t-3)e^{2(t-3)}$.
   - The transform of e^{2t} is $1/(s-2)$.
   - Applying the theorem: $\mathcal{L}\{f(t)\} = e^{6}e^{-3s}\frac{1}{s-2}$.

   Answer: \frace^6-3ss-2

2. Solve the IVP $y' + 2y = u(t-1)$ with $y(0) = 0$.
   - Transform the equation: $(s+2)Y(s) = e^{-s}/s$.
   - Solve for $Y(s)$: $Y(s) = e^{-s}\frac{1}{s(s+2)}$.
   - Partial fraction decomposition of $G(s) = \frac{1}{s(s+2)} = \frac{1}{2}(\frac{1}{s} - \frac{1}{s+2})$.
   - Inverse transform of $G(s)$ is $g(t) = \frac{1}{2}(1 - e^{-2t})$.
   - Apply the shift: $y(t) = u(t-1)g(t-1) = \frac{1}{2}u(t-1)(1 - e^{-2(t-1)})$.

   Answer: \frac12 u(t-1) (1 - e^-2(t-1))

3. Find the inverse Laplace transform of $H(s) = \frac{e^{-2s}}{s^2+9}$.
   - Identify $G(s) = \frac{1}{s^2+9}$.
   - The inverse transform of $G(s)$ is $g(t) = \frac{1}{3}\sin(3t)$.
   - The factor e^{-2s} indicates a shift of $a = 2$.
   - Apply the second shifting theorem: $h(t) = u(t-2)g(t-2)$.
   - Substitute $t-2$ into $g(t)$: $h(t) = \frac{1}{3}u(t-2)\sin(3(t-2))$.

   Answer: \frac13 u(t-2) \sin(3(t-2))
